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Hybrid and Multiplicative Overlapping Schwarz Algorithms with Standard Coarse Spaces for Mixed Linear Elasticity and Stokes Problems

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Abstract. The goal of this work is to construct and study hybrid and multiplicative two-level overlapping Schwarz algorithms with standard coarse spaces for the almost incompressible linear elasticity and Stokes systems, discretized by mixed finite and spectral element methods with discontinuous pressures. Two different approaches are considered to solve the resulting saddle point systems: a) a preconditioned conjugate gradient (PCG) method applied to the symmetric positive definite reformulation of the almost incompressible linear elasticity system obtained by eliminating the pressure unknowns; b) a GMRES method with indefinite overlapping Schwarz preconditioner applied directly to the saddle point formulation of both the elasticity and Stokes systems. Condition number estimates and convergence properties of the proposed hybrid and multiplicative overlapping Schwarz algorithms are proven for the positive definite reformulation of almost incompressible elasticity. These results are based on our previous study [8] where only additive Schwarz preconditioners were considered for almost incompressible elasticity. Extensive numerical experiments with both finite and spectral elements show that the proposed overlapping Schwarz preconditioners are scalable, quasi-optimal in the number of unknowns across individual subdomains and robust with respect to discontinuities of the material parameters across subdomains interfaces. The results indicate that the proposed preconditioners retain a good performance also when the quasi-monotonicity assumption, required by the available theory, does not hold.

AMS subject classifications: 65M55, 65N22, 65N30, 65F08, 65M22

Key words: Overlapping Schwarz preconditioners, almost incompressible linear elasticity, Stokes equations, saddle point problems, finite and spectral elements.

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1 Introduction

Finite and spectral element discretizations of the linear elasticity system suffer increasingly from locking effects and ill-conditioning, when the material approaches the incompressible limit, if only the displacement variables are used. One remedy to overcome the locking effect consists in using the following mixed form of the linear elasticity operator

$$\begin{bmatrix} -2\mu \mathrm{div}\,\mathbf{D}(\cdot) & \mathrm{grad} \\ -\mathrm{div} & -\frac{1}{\lambda}I \end{bmatrix}. \tag{1.1}$$

Here, λ and μ are the Lamé constants, expressed as

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)} \tag{1.2}$$

with E being the modulus of elasticity (Young's modulus) and ν being the Poisson ratio of the elastic material. It is easy to see that the above mixed operator degenerates to the incompressible Stokes operator when the $(2,2)$ block is zero. Finite and spectral element discretization of this mixed formulation lead to large saddle point systems whose iterative solution requires effective and efficient preconditioners. The goal of this paper is to construct and study hybrid and multiplicative two-level overlapping Schwarz preconditioners with standard coarse spaces for the mixed discretization of almost incompressible linear elasticity and Stokes systems. Earlier works on overlapping Schwarz methods for linear elasticity have focused on the compressible case in which the Poisson ratio ν is bounded away from $1/2$. The related recent developments are as follows: some nonoverlapping domain decomposition algorithms for mixed elasticity and Stokes systems have studied Wirebasket and Balancing Neumann-Neumann methods, see [1, 19, 34, 35], and FETI-DP and BDDC methods for the incompressible limit, see [20, 22, 25, 26, 36, 38]; more recent applications include fluid-structure interaction [2], computational fluid dynamics [15, 20], and isogeometric analysis [3].

We will consider two different approaches to solve the resulting saddle point systems: a) a preconditioned conjugate gradient (PCG) method applied to the symmetric positive definite reformulation of the almost incompressible linear elasticity system obtained by eliminating the pressure unknowns element by element; b) a GMRES method with indefinite overlapping Schwarz preconditioner applied directly to the saddle point formulation of both the elasticity and Stokes systems. In both approaches, our main interest is the almost incompressible case and the incompressible Stokes limit.

Earlier works proposed some additive and hybrid two-level overlapping Schwarz algorithms in which exotic coarse spaces are used in the designing of the coarse grid problem, see [11, 12]. The tools and theoretical analysis developed in these papers yield a condition number bound on the preconditioned additive Schwarz operator which is cubic (in contrast, it is linear for the compressible case) in the relative overlap size and which grows logarithmically with the number of elements across individual subdomains.