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Nonconforming Finite Element Method Applied to the Driven Cavity Problem

Roktaek Lim[1] and Dongwoo Sheen[2,*]

[1] *Division of Mathematical Sciences, School of Physical & Mathematical Sciences, Nanyang Technological University, 21 Nanyang Link, Singapore 637371.*
[2] *Department of Mathematics, Seoul National University, Seoul 08826, Korea.*

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Abstract. A cheapest stable nonconforming finite element method is presented for solving the incompressible flow in a square cavity without smoothing the corner singularities. The stable cheapest nonconforming finite element pair based on $P_1 \times P_0$ on rectangular meshes [29] is employed with a minimal modification of the discontinuous Dirichlet data on the top boundary, where $\widetilde{\mathscr{P}_0^h}$ is the finite element space of piecewise constant pressures with the globally one-dimensional checker-board pattern subspace eliminated. The proposed Stokes elements have the least number of degrees of freedom compared to those of known stable Stokes elements. Three accuracy indications for our elements are analyzed and numerically verified. Also, various numerous computational results obtained by using our proposed element show excellent accuracy.

AMS subject classifications: 65N30, 74S05

Key words: Nonconforming finite element method, incompressible Navier-Stokes equations, lid driven cavity problem.

1 Introduction

The lid driven square cavity has been one of the most popular benchmark problems for new numerical methods for the incompressible Navier-Stokes equations in terms of accuracy, numerical efficiency and so on. To refer only few see [4,8,17,18], for instance, and the references therein. The presence of singularities at the upper corners of the cavity is the source of numerical difficulties for solving the cavity flow problem. It is usually erroneous to use high-order methods without handling the corner singularities due to

*Corresponding author. *Email addresses:* rokt.lim@gmail.com (R. Lim), dongwoosheen@gmail.com (D. Sheen)

the Gibbs phenomenon. Many studies have been carried out to overcome this difficulty. Barragy and Carey [6] used a p-version finite element formulation ($p \geq 6$) combined with a strongly graded and refined mesh to handle the corner singularities. Other studies change the boundary condition to overcome this difficulty: see, for instance, [20,21,34,36], and the references therein. The latter approach are coined as the so-called regularized lid driven cavity problem. The constant boundary condition for velocity is replaced by a function that vanishes at the upper corners of cavity [20,36]. Botella and Peyret [8] solved a regularized cavity problem by using a subtraction method of the leading terms from the asymptotic expansion of the solution of the Navier-Stokes equations in the vicinity of the corners, where the velocity is discontinuous. Sahin and Owens [34] inserted leaks across the heights of the finite volumes at the corners between the lid and the vertical walls to handle the corner singularities. Many studies reported that in the critical Reynolds number range [7000,8500] Hopf bifurcations occur for the lid driven square cavity problem [4,18,20,36]. Bruneau and Saad [9] revisited the issue of bifurcation using third-order time discretization schemes with 512×512 finite difference spatial discretizations. They observed the first bifurcation occurs between $Re = 8000$ and $Re = 8050$. Guermond and Minev [24] reported three-dimensional benchmark solutions using a direction splitting method introduced in [22, 23]. They also provided two-dimensional solutions, which are correct up to at least three digits, for $Re = 1000$ using the uniform 5000×5000 MAC stencil. Instead of the square domain, Glowinski *et al.* [21] considered a semi-circular cavity-driven flow with a special time-dependent regularization on the Dirichlet data at the two corners: they observed Hopf bifurcations around $Re=6600$, which is smaller than the case of square domain, using an iso-parametric variant of the Bercovier-Pironneau element [7] introduced in [20].

The purpose of the current paper is to try to solve the lid driven square cavity problem without any regularization at the corners, employing nonconforming finite element pairs whose degrees of freedom and implementation are as cheap as possible. As the nonconforming elements use the values at the midpoints of edges as DOFs, instead of those at the vertices, the discontinuity singularities at the corners are naturally treated without any regularization. Our nonconforming finite element pairs are based on the two stable nonconforming finite element pairs on uniform square meshes [29] introduced for the stationary incompressible Stokes problem. The two pairs are briefly described as follows: The first of them uses the P_1-nonconforming quadrilateral element [32] for the approximation of the velocity field, componentwise, while the pressure is approximated by a subspace of the piecewise constant functions whose dimension is two less than the number of squares in the mesh. The second of them is a one-dimensional modification of the above finite element pairs to both velocity and pressure spaces: the velocity space is enriched by a globally one-dimensional DSSY(Douglas-Santos-Sheen-Ye)-type bubble function [11, 15, 26] while the pressure space is the subspace of the piecewise constant functions whose dimension is one less than the number of squares in the mesh in order to fulfill the mean-zero property. The stability and optimal convergence results for these element pairs applied to the stationary Stokes equations with homogeneous Dirichlet