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Pressure-Correction Projection FEM for Time-Dependent Natural Convection Problem

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Abstract. Pressure-correction projection finite element methods (FEMs) are proposed to solve nonstationary natural convection problems in this paper. The first-order and second-order backward difference formulas are applied for time derivative, the stability analysis and error estimates of the semi-discrete schemes are presented using energy method. Compared with characteristic variational multiscale FEM, pressure-correction projection FEMs are more efficient and unconditionally energy stable. Ample numerical results are presented to demonstrate the effectiveness of the pressure-correction projection FEMs for solving these problems.

AMS subject classifications: 65N15, 65N30, 76D07, 65N12, 35Q79

Key words: Time-dependent natural convection problems, pressure-correction projection FEMs, backward difference formula, stability analysis, error estimate.

1 Introduction

We consider in this paper numerical schemes for solving the time-dependent nonlinear natural convection (NC) equations:

$$\begin{cases} \mathbf{u}_t - Pr\,\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = Pr\;Ra\,\mathbf{j}T & \text{in } \Omega\times(0,T_1], \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega\times(0,T_1], \\ T_t - \kappa\Delta T + (\mathbf{u}\cdot\nabla)T = \gamma & \text{in } \Omega\times(0,T_1], \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}^0, \quad T(\mathbf{x},0) = T^0 & \text{in } \Omega\times\{0\}, \\ \mathbf{u} = 0, \quad T = 0 & \text{on } \partial\Omega\times(0,T_1], \end{cases} \tag{1.1}$$

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where $\mathbf{x}=(x_1,x_2)$, $\mathbf{j}=(0,1)^T$, Ω is an open bounded domain in $\mathbb{R}^2$ with a sufficiently smooth boundary $\partial\Omega$, Pr,κ, Ra, T_1 and γ represent the Prandtl number, the thermal conductivity parameter, the Rayleigh number, the given final time and the forcing function, respectively. The unknown functions are velocity vector $\mathbf{u}=(u_1,u_2)$, pressure p and temperature T. Just as [10,17] presented in Navier-Stokes equations, we consider a homogeneous Dirichlet boundary conditions in the above for simplicity and to fix the idea.

The NC equations constitute an important system of equations in atmospheric dynamics and dissipative nonlinear system of equations including continuity equation, momentum equation and energy equation. Since the system of equations contains the velocity field, the pressure as well as the temperature field, it is a challenge to construct efficient and unconditionally energy stable schemes. While an enormous body of works have been devoted to developing efficient schemes for the stationary or nonstationary NC equations (cf. [1,13,18,19,23,24] and the references therein), much less attention has been paid to developing unconditionally energy stable schemes. Boland and Layton [1] gave some numerical analysis and numerical results for the non-stationary natural convection equations. Luo and his collaborators offered lowest order finite difference scheme based on mixed finite element method (FEM) for non-stationary natural convection problem in [13]. In addition, Si and his collaborators [18] formulated the modified characteristics Gauge-Uzawa FEM for time dependent conduction-convection problems. Wu et al. [23] studied a characteristic variational multiscale (C-VMS) FEM for time-dependent conduction-convection problem. The objective for this paper is to design efficient and unconditionally energy stable numerical schemes for the coupled system.

For the above coupled nonlinear system, there are some numerical difficulties: (i) the coupling of the velocity and pressure through the incompressibility constraint; (ii) the presence of nonlinear terms; (iii) the coupling of flow field and temperature field. However, the first is the main difficulty. To overcome this difficulty, a common strategy to decouple the computation of the pressure from the velocity is to use a projection-type scheme as in the case for Navier-Stokes equations (cf., [2,7,20]).

Projection methods can be viewed as fractional/splitting step methods, where the convection-diffusion and the incompressibility are dealt with in two different sub-steps (see [4,5,7–10,14,16]). For pressure-correction (PC) projection methods, the pressure is made explicit in the first substep and is corrected in the second one by projecting the provisional velocity onto the space of incompressible vector fields. The velocity obtained in the convection-diffusion sub-step is projected in order to satisfy the weak incompressibility condition. As we know, standard pressure-correction (SPC) projection scheme which may be first given by Goda [6] (also see [7,16,17]) suffers from the nonphysical pressure boundary condition that induces a numerical boundary layer, which degrades the accuracy of the pressure approximation. Timmermans et al. [21] proposed the rotational pressure-correction (RPC) scheme (also see [7,10,17]) that leads to improved pressure approximation. More importantly and appealing, using projection methods, one only needs to solve a sequence of decoupled elliptic equations for the velocity and the pressure at each time step, making it very efficient for large scale numerical simulations. As