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A Simple Explanation of Superconvergence for Discontinuous Galerkin Solutions to $u_t + u_x = 0$

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Abstract. The superconvergent property of the Discontinuous Galerkin (DG) method for linear hyperbolic systems of partial differential equations in one dimension is explained by relating the DG method to a particular continuous method, whose accuracy depends in part on a local analysis, and in part on information transferred from upwind elements.

AMS subject classifications: 35L03, 65M60

Key words: Linear advection, discontinuous Galerkin, superconvergence.

1 Introduction

Out of all the high-order methods devised for solving hyperbolic conservation laws, the one that is presently thought by many to be closest to achieving practical utility is the discontinuous Galerkin (DG) method. An intriguing property of this scheme is that when it is applied to some simple problems it is observed to achieve [6] an unexpected order of accuracy, namely h^{2p+1}, where h is the grid spacing and p is the polynomial degree used in the reconstruction. This "superconvergent accuracy" is much better than the accuracy h^{p+1} usually associated with finite difference methods based on polynomial interpolation. It was proved by Ainsworth [2] that the dispersion relationship of the DG method is accurate to this order, and pointwise superconvergence of the solution at the Radau points was shown by Adjerid *et al.* [1] and by Yang and Shu [9]. Proof that superconvergence occurs for the element averages was only recently given by Cao, Zhang and Zou [4].

In the present paper, a simple proof is given of a stronger result, that the DG method is superconvergent everywhere, if a particular continuous reconstruction is employed.

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This insight provided by this proof may facilitate the extension of superconvergence to a wider class of problems.

We will several times have recourse to the following well-known lemma:

If a discrete version of some differential operator is exact whenever the input is a polynomial function of degree not exceeding q, then its order of accuracy is at least q.

This follows from realising that the Taylor series of the truncation error must vanish to order q.

2 The (r,s) method

We will solve the linear advection problem,

$$\partial_t u+\partial_x u=0, \tag{2.1}$$

on a sequence of equally-spaced elements (for simplicity) $\mathcal{E}_k: x\in[x_k\pm\frac{h}{2}]$, $x_k=x_0+kh$. We will study schemes that assign a number r of d.o.f ($\mathbf{u}^k=u_1^k,\cdots,u_r^k$) to each element. We call these the intrinsic d.o.f.. A number s of independent d.o.f. $\mathbf{v}^{k+1/2}=(v_1^{k+1/2}\cdots v_s^{k+1/2})$ are shared between elements $\mathcal{E}_k$ and $\mathcal{E}_{k+1}$ and are called the shared d.o.f. These may be associated with the interface $k+1/2$ or with an overlap region.

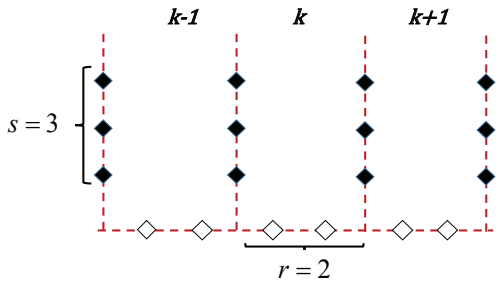

Figure 1: Schematic storage locations for the (r,s) method.

For each cell $\mathcal{E}_k$ the solution can be reconstructed as a polynomial of degree $r+2s-1$ using the d.o.f. contained in $\mathbf{v}^{k-1/2},\mathbf{u}^k,\mathbf{v}^{k+1/2}$

$$u_k(x)=\sum_{m=1}^{m=r+2s} w_m(t)\phi_m(x) \tag{2.2}$$

using a local counting variable m to include all the d.o.f. intrinsic to or shared by $\mathcal{E}_k$. Each d.o.f. is associated with a basis function $\phi_m(x)$, where the set of basis functions spans the polynomials in x of degree $r+2s-1$ or less. A discrete version of (2.1) for that element is the residual

$$r^k(x)=\sum_{m=1}^{m=r+2s} \dot{w}_m^k\phi_m^k(x)+w_m^k\partial_x\phi_m^k(x)\simeq 0, \tag{2.3}$$