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Schemes with Well-Controlled Dissipation. Hyperbolic Systems in Nonconservative Form

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Abstract. We propose here a class of numerical schemes for the approximation of weak solutions to nonlinear hyperbolic systems in nonconservative form — the notion of solution being understood in the sense of Dal Maso, LeFloch, and Murat (DLM). The proposed numerical method falls within LeFloch-Mishra's framework of schemes with well-controlled dissipation (WCD), recently introduced for dealing with small-scale dependent shocks. We design WCD schemes which are consistent with a given nonconservative system at arbitrarily high-order and then analyze their linear stability. We then investigate several nonconservative hyperbolic models arising in complex fluid dynamics, and we numerically demonstrate the convergence of our schemes toward physically meaningful weak solutions.

AMS subject classifications: 35L65, 65M06, 76N15

Key words: Nonlinear hyperbolic, nonconservative system, discrete scheme, controlled dissipation, complex fluid dynamics.

1 Introduction

This is a follow-up to the papers [17, 32], in which two of the authors have introduced a class of schemes, referred to as the *schemes with well-controlled dissipation* (WCD), which allow one to compute, with robustness and accuracy, small-scale dependent shock

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wave solutions to nonlinear hyperbolic systems of conservation laws. We do not attempt to review the problem of nonclassical shocks and the corresponding numerical techniques and, instead, we refer to the textbook by LeFloch [29] together with the recent review by LeFloch and Mishra [32] and references therein.

In the present paper, we investigate nonlinear hyperbolic models in nonconservative form, such as those arising in the modeling of two-phase (liquid-vapor) flows and shallow water (two-layer) flows. Shock waves appear often in these contexts and developing reliable numerical methods is an essential challenge. Specifically, we consider nonlinear hyperbolic systems in one spatial variable, of the general form

$$\begin{aligned} &U_t+A(U)U_x=0, \qquad U=U(t,x)\in\mathbb{R}^N,\\ &U(0,x)=U_0(x), \end{aligned} \tag{1.1}$$

where U is the vector of primitive unknowns and $A=A(U)$ is a smooth map defined on (a subset of) $\mathbb{R}^N$. We assume that, for each relevant U, the matrix $A(U)$ admits distinct real eigenvalues denoted by $\lambda_1<\lambda_2<\cdots<\lambda_N$ and, therefore, a corresponding basis of right-eigenvectors, denoted by $r_j(U)_{1\leq j\leqslant N}$. We are interested in (possibly) discontinuous solutions to (1.1) that can be realized as (singular) limits of (smooth) solutions to systems with vanishing diffusion and (possibly) dispersion, that is,

$$\begin{aligned} &U_t+A(U)U_x=\epsilon\big(B(U)U_x\big)_x+\alpha\epsilon^2(C_1(U)(C_2(U)U_x)_x)_x,\\ &U(0,x)=U_0(x). \end{aligned} \tag{1.2}$$

The challenging question of defining weak solutions to (1.1) was solved around 1990 by LeFloch and his collaborators. Recall that it was first proposed in [25] to rely on Volpert's product and in [26] to rely on traveling wave profiles associated with an augmented model taking into account small-scale effects. A general notion of nonconservative product was then introduced by Dal Maso, LeFloch and Murat (in the preprint included in [27], later published in [15]) who also solved the Riemann problem for nonconservative systems. Recall also that the existence of weak solutions in the DLM sense to the Cauchy problem was established for nonlinear hyperbolic systems (1.1) by LeFloch and Liu [31] via Glimm's random choice method when the initial data has small total variation. The existence of traveling wave solutions to nonconservative systems was established later on in [38]; see also [2].

The numerical investigation of nonconservative hyperbolic problems was initiated in Hou and LeFloch [22], motivated by an earlier study by Karni [24] for the system of gas dynamics equations. The class of nonconservative systems and, more generally, systems admitting small-scale dependent shock waves, leads to particularly challenging problems as was first studied by LeFloch and collaborators [14,21–23,33,34] and later further investigated in [1,8,37]. Recall that a variety of approaches for nonconservative systems were proposed in the literature, especially by Gosse [20], Berthon and Coquel [3,5,6] together with LeFloch [7], Parés and collaborators [11,12,35,36], Mishra and collaborators [9,19], and Berthon, Boutin, and Turpault [4].