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# Numerical Study of Partially Conservative Moment Equations in Kinetic Theory

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**Abstract.** Moment models are often used for the solution of kinetic equations such as the Boltzmann equation. Unfortunately, standard models like Grad's equations are not hyperbolic and can lead to nonphysical solutions. Newly derived moment models like the Hyperbolic Moment Equations and the Quadrature-Based Moment Equations yield globally hyperbolic equations but are given in partially conservative form that cannot be written as a conservative system.

In this paper we investigate the applicability of different dedicated numerical schemes to solve the partially conservative model equations. Caused by the non-conservative type of equation we obtain differences in the numerical solutions, but due to the structure of the moment systems we show that these effects are very small for standard simulation cases. After successful identification of useful numerical settings we show a convergence study for a shock tube problem and compare the results to a discrete velocity solution. The results are in good agreement with the reference solution and we see convergence considering an increasing number of moments.

**AMS subject classifications**: 35Q20, 35Q35, 35L02

**Key words**: Kinetic theory, moment method, hyperbolicity, non-conservative numerics.

## 1 Introduction

Numerical simulations of applications involving rarefied gases require the solution of the Boltzmann transport equation (BTE), especially if moderate to large Knudsen numbers are considered and particle methods like Direct Simulation Monte Carlo (see [3]) on the one hand are still too costly whereas standard fluid dynamics equations on the other hand are not accurate enough to describe possible non-equilibrium effects like the Knudsen paradox (see [20]).

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Moment methods are one possibility to efficiently solve the BTE as relatively few variables (usually the so-called *moments*) are used to describe the flow solution. The first example is Grad's method (see [8]) which uses an expansion of the unknown distribution function around local equilibrium in Hermite polynomials and results in a system of equations for the coefficients of this expansion, also called *moment system*. However, it has been shown in [4] that the resulting PDE system of Grad's method looses hyperbolicity for larger deviations from equilibrium, resulting in nonphysical infinite or imaginary propagation wave speeds. Hyperbolicity is a necessary condition for stable solutions of a moment system (see also [24]).

The maximum entropy approach proposed by Levermore (see e.g. [15]) has good approximation properties and is hyperbolic by construction, but unfortunately does not have an explicit flux formulation. Instead it requires a costly optimization which makes this method difficult to apply for a large number of moments. An ansatz based on multivariate Pearson-IV-Distributions was proposed by Torrilhon in [22] but is difficult to generalize to the case with more higher order moments.

Following the work in [4] and [10], a new framework for the derivation of hyperbolic moment models has been derived in [7]. The proposed operator projection framework can be seen as a generalization of Grad's method with additional modification of the equations such that the resulting moment system is globally hyperbolic. Two examples are the hyperbolic moment equations (HME, first derived in [4]) and the quadrature-based moment equations (QBME, first derived in [11]). Both methods yield very similar systems of equations as exemplified for a simple test case in [12]. In comparison to Grad's standard model, only some of the higher order equations are changed. Both models exhibit the same physical adaptivity as can be found either in the original model by Grad or due to the transformation of the Boltzmann equation by Kauf in [9].

Despite the beneficial hyperbolicity, a major drawback of the novel equations is the lack of a conservative form. Due to small changes in the last equations, the new HME and QBME systems are in so-called *partially conservative* form, which means that only a subset of the whole system can be written in conservative form. One or two of the last equations are not in conservative form which makes the application of standard finite volume schemes impossible. In this paper we therefore apply non-conservative numerical schemes to solve the system, following the theory in [16].

Various analytical properties of HME and QBME have been extensively studied in the papers mentioned before. The purpose of this paper is the investigation and justification of different numerical schemes to solve the partially conservative PDE system. It is important to study the performance of different schemes first, before addressing model accuracy itself because the solution of non-conservative systems is much more difficult than in the standard conservative case and different schemes can converge to different numerical solutions. It is our aim to investigate these problems and eventually identify the best scheme. Afterwards, we want to use this scheme to get more numerical results for the new moment systems. Consequently, the tests with different schemes will be followed by a study of the model performance with respect to accuracy and convergence