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High Order Well-Balanced Weighted Compact Nonlinear Schemes for Shallow Water Equations

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Abstract. In this study, a numerical framework of the high order well-balanced weighted compact nonlinear (WCN) schemes is proposed for the shallow water equations based on the work in [S. Zhang, S. Jiang, C.-W Shu, J. Comput. Phys. 227 (2008) 7294-7321]. We employ a special splitting technique for the source term proposed in [Y. Xing, C.-W Shu, J. Comput. Phys. 208 (2005) 206-227] to maintain the exact C-property, which can be proved theoretically. In the meantime, the genuine high order accuracy of the numerical scheme can be observed successfully, and small perturbation of the stationary state can be resolved and evolved well. In order to capture the strong discontinuities and large gradients, the fifth-order upwind weighted nonlinear interpolations together with the fourth/sixth order cell-centered compact scheme are used to construct different WCN schemes. In addition, the local characteristic projections are considered to further restrain the potential numerical oscillations. A variety of representative one- and two-dimensional examples are tested to demonstrate the good performance of the proposed schemes.

AMS subject classifications: 35L65, 35L67

Key words: Shallow water equations, C-property, weighted compact nonlinear scheme, source term.

1 Introduction

The shallow water equations are usually a system of hyperbolic conservation laws with additional source terms that describes geophysical flows, especially when the horizontal length scale is much greater than the vertical length scale. The equations play an important role in the modeling and simulation of free surface flows in rivers and coastal

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areas, and can predict tides, storm surge levels and coastline changes from hurricanes and ocean currents [33].

The two-dimensional shallow water equations can be written as

$$\frac{\partial h}{\partial t}+\frac{\partial (hu)}{\partial x}+\frac{\partial (hv)}{\partial y}=0, \tag{1.1a}$$

$$\frac{\partial (hu)}{\partial t}+\frac{\partial (hu^2+\frac{1}{2}gh^2)}{\partial x}+\frac{\partial (huv)}{\partial y}=-ghb_x, \tag{1.1b}$$

$$\frac{\partial (hv)}{\partial t}+\frac{\partial (huv)}{\partial x}+\frac{\partial (hv^2+\frac{1}{2}gh^2)}{\partial y}=-ghb_y, \tag{1.1c}$$

where b is the vertical height of the bottom topography, from an arbitrary level of reference, h is the water depth above the bottom topography, (u,v) is the velocity vector, and $g=9.812$ is the gravitational constant. Due to the practical importance, this system has been theoretically and numerically studied for many years [6,11,12,16,17,21,30,32–34]. As we know, this system admits stationary solutions in which the nonzero flux gradients can be exactly balanced by the source terms in the steady state case. In other words, a good numerical scheme should be capable of preserving the stationary solution $h+b=constant$ and $(u,v)=(0,0)$. Therefore, Vukovic and Sopta [32] made an important modification on the classical weighted essentially non-oscillatory (WENO) scheme [14] and applied it to the shallow water equations for preserving the balance between the flux gradient and source terms. The scheme was verified for maintaining the exact conservation property (C-property) [2,31], when applied to a quiescent flow, where b is a smooth or non-smooth topography of the sea floor. Then the exact C-property was demonstrated in the one-dimensional sediment transport equations [5]. In [27], Robers et al. proved the balancing between the flux gradient and the source term. Xing and Shu [34] proposed the high order well-balanced WENO scheme, which maintained the exact C-property and achieved genuine high order accuracy for the general solutions of the shallow water equations. However, numerical experiments show that the classical WENO scheme is too dissipative for complex flow structures such as turbulence [15]. In [11,35,36], the well-balanced Runge-Kutta discontinuous Galerkin methods for the shallow water equations were also proposed respectively in the last decade. We refer to [33] and references therein for more detailed descriptions on the well-balanced schemes.

To increase the resolution and decrease the dissipation of the simulations, the well-balanced hybrid upwind-WENO [21] and compact-WENO [39] schemes were designed respectively in solving the shallow water equations. In the hybrid schemes, how to generate a high quality shock indicator for discontinuities in a complex flow is nontrivial. It is an active research area [10, 26], but out of the scope of this paper. Moreover, based on the idea of WENO scheme, a class of weighted compact nonlinear (WCN) schemes were developed [8, 9] which has better wave resolution and similar ability to capture discontinuities as the classical WENO scheme. Then it has been extended to the ninth order [22, 25, 37] which demonstrates the increasing resolutions with the increasing orders.