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A Moving Mesh Finite Difference Method for Non-Monotone Solutions of Non-Equilibrium Equations in Porous Media

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Abstract. An adaptive moving mesh finite difference method is presented to solve two types of equations with dynamic capillary pressure effect in porous media. One is the non-equilibrium Richards Equation and the other is the modified Buckley-Leverett equation. The governing equations are discretized with an adaptive moving mesh finite difference method in the space direction and an implicit-explicit method in the time direction. In order to obtain high quality meshes, an adaptive monitor function with directional control is applied to redistribute the mesh grid in every time step, then a diffusive mechanism is used to smooth the monitor function. The behaviors of the central difference flux, the standard local Lax-Friedrich flux and the local Lax-Friedrich flux with reconstruction are investigated by solving a 1D modified Buckley-Leverett equation. With the moving mesh technique, good mesh quality and high numerical accuracy are obtained. A collection of one-dimensional and two-dimensional numerical experiments is presented to demonstrate the accuracy and effectiveness of the proposed method.

AMS subject classifications: 35C07, 35Q35, 65M50, 74S20, 76S05

Key words: Relaxation non-equilibrium Richards equation, modified Buckley-Leverett equation, saturation overshoot, traveling wave analysis, moving mesh finite difference method.

1 Introduction

For the past several decades, since the observations of saturation overshoot and gravity driven fingers [1–4], there have been a great deal of experimental and theoretical studies on the mechanism and modeling of such phenomena. Stauffer [5], Hassanizadeh

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and Gray [6], Kalaydjian et al. [7] proposed a dynamic (non-equilibrium) relationship between capillary pressure and saturation to explain the occurrence of non-monotone saturation and capillary pressure when water is injected into initially dry sandy porous media. Eliassi and Glass investigated three additional forms referring to as a hypodiffusive form, a hyperbolic form and a mixed form in [8], they obtained saturation overshoot successfully by using the hypodiffusive form [9]. Nieber et al. [10], Chapwanya and Stockie [11] investigated the gravity-driven fingers by supplementing the Richards equation with the dynamic capillary pressure-saturation relationship, as well as including hysteretic effects. Their results demonstrate that the non-equilibrium Richards equation is capable of reproducing realistic fingers for a wide range of physically relevant parameters. Inspired by fingering instabilities in the flow of thin films, Cueto-Felgueroso and Juanes [12] put forward a phase field model using the idea of including the effect of a macroscopic interface in the mathematical description of unsaturated flow. Their model predictions agreed well with the lab measurements [4]. In the above mentioned references, most of models can be described as extensions to the Richards equation, besides, other approaches characterizing the saturation overshoot have also been investigated. Refs. [13,14] studied a generalized theory by introducing percolating and non-percolating fluid phases into a traditional mathematical model. DiCarlo et al. [15] developed a multiphase, fractional flow approach to describe the physics behind the displacement front that includes the viscosity of the gas. Refs. [16–18] simulated saturation overshoot by incorporating the dynamic capillary pressure with a traditional fractional flow equation. Their results suggest that the non-equilibrium fractional flow equation has the ability to model saturation overshoot.

Among the proposed theories, two models incorporating the dynamic capillary pressure relationship have attracted considerable interest in recent years. One is the relaxation non-equilibrium Richards equation (RNERE), and the other is the modified Buckley-Leverett equation (MBLE). Results on stability, traveling wave (TW) solutions, global existence, phase plane analysis and uniqueness of weak solutions are given in [19–24]. Numerical simulations [10, 11, 21, 25, 26] of the RNERE and the MBLE show that with appropriate parameters, both models will generate non-monotonic distribution of saturation, and the RNERE can become unstable in 2D when the flow profiles are sufficiently non-monotonic [20] which agree with the TW analysis and stability result.

In order to numerically solve these non-equilibrium equations, a variety of numerical methods have been developed in literature. Peszynska and Yi [27] proposed a cell-centered finite difference method and a locally conservative Eulerian-Lagrangian method, but they noticed that such methods may cause instabilities in convection-dominated cases and for large dynamic effects. A finite difference method which combined a minmod slope limiter based on the first order upwind and Richtmyer's schemes was used by van Duijn et al. [21]. The solutions obtained by this scheme agreed well with the TW results. Wang and Kao [28] extended the second and third order central schemes to capture the nonclassical solutions of the MBLE. Kao et al. [25] split the MBLE into a high-