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Curious Convergence Properties of Lattice Boltzmann Schemes for Diffusion with Acoustic Scaling

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Abstract. We consider the D1Q3 lattice Boltzmann scheme with an acoustic scale for the simulation of diffusive processes. When the mesh is refined while holding the diffusivity constant, we first obtain asymptotic convergence. When the mesh size tends to zero, however, this convergence breaks down in a curious fashion, and we observe qualitative discrepancies from analytical solutions of the heat equation. In this work, a new asymptotic analysis is derived to explain this phenomenon using the Taylor expansion method, and a partial differential equation of acoustic type is obtained in the asymptotic limit. We show that the error between the D1Q3 numerical solution and a finite-difference approximation of this acoustic-type partial differential equation tends to zero in the asymptotic limit. In addition, a wave vector analysis of this asymptotic regime demonstrates that the dispersion equation has nontrivial complex eigenvalues, a sign of underlying propagation phenomena, and a portent of the unusual convergence properties mentioned above.

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1 Introduction

Lattice Boltzmann models are simplifications of the continuum Boltzmann equation obtained by discretizing in both physical space and velocity space. The discrete velocities v_i retained typically correspond to lattice vectors of the discrete spatial lattice. That is, each lattice vertex x is linked to a finite number of neighboring vertices by lattice vectors $v_i \Delta t$. A particle distribution f is therefore parametrized by its components in each of the discrete velocities, the vertex x of the spatial lattice, and the discrete time t. A time step of a classical lattice Boltzmann scheme [11] then contains two steps:

(i) a relaxation step where distribution f at each vertex x is locally modified into a new distribution f^*, and

(ii) an advection step based on the method of characteristics as an exact time-integration operator. We employ the multiple-relaxation-time approach introduced by d'Humières [10], wherein the local mapping $f \longmapsto f^*$ is described by a nonlinear diagonal operator in a space of moments, as detailed in Section 2.

In [5], we have studied the asymptotic expansion of various lattice Boltzmann schemes with multiple-relaxation times for different applications. We used the so-called acoustic scaling, in which the ratio $\lambda \equiv \Delta x / \Delta t$ is kept fixed. In this manner, we demonstrated the possibility of approximating diffusion processes described by the heat equation.

In his very complete work, Dellacherie [3] has described unexpected results in simulations for advection-diffusion processes. In this contribution, we endeavor to explain those results by studying the convergence of the D1Q3 lattice Boltzmann scheme when we try to approximate a pure diffusion process.

We begin this paper by recalling some fundamental algorithmic aspects of the D1Q3 lattice Boltzmann scheme in Section 2. Then, in Section 3 we describe a first illustrative numerical experiment. In Section 4 we present a new convergence analysis, followed by another numerical experiment in Section 5, in which the D1Q3 lattice Boltzmann scheme is studied far from the usual values of its parameters. Finally, a wave vector analysis is proposed in Section 6.

2 Diffusive D1Q3 lattice Boltzmann scheme

In this work, we consider the so-called D1Q3 lattice Boltzmann scheme in one spatial dimension. The spatial step $\Delta x > 0$ is given, and each node x is an integer multiple of this spatial step : $x \in \mathbb{Z}\Delta x$. The time step $\Delta t > 0$ is likewise given, and each discrete time t is an integer multiple of Δt. We adopt so-called acoustic scaling (see e.g., [12]), so the numerical velocity associated with the mesh,

$$\lambda \equiv \frac{\Delta x}{\Delta t} \tag{2.1}$$