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A Uniformly Convergent Scheme for Radiative Transfer Equation in the Diffusion Limit Up to the Boundary and Interface Layers

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Abstract. In this paper, we present a numerical scheme for the steady-state radiative transfer equation (RTE) with Henyey-Greenstein scattering kernel. The scattering kernel is anisotropic but not highly forward peaked. On the one hand, for the velocity discretization, we approximate the anisotropic scattering kernel by a discrete matrix that can preserve the diffusion limit. On the other hand, for the space discretization, a uniformly convergent scheme up to the boundary or interface layer is proposed. The idea is that we first approximate the scattering coefficients as well as source by piecewise constant functions, then, in each cell, the true solution is approximated by the summation of a particular solution and a linear combinations of general solutions to the homogeneous RTE. Second-order accuracy can be observed, uniformly with respect to the mean free path up to the boundary and interface layers. The scheme works well for heterogenous medium, anisotropic sources as well as for the strong source regime.

AMS subject classifications: 65L12, 76N20, 35Q70

Key words: Radiative transfer equation, discrete Ordinates method, tailored finite point method, anisotropic scattering, asymptotic-preserving.

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1 Introduction

To monitor the development of disease-associated processes on a molecular or gene level prior to the appearance of macroscopic tissue changes, a molecular imaging technique has been developed [15]. A fluorescent bio-chemical marker that emits light is injected into a biological system to map the distribution of the administered molecule or gene. A highly sensitive camera is then used to capture the bioluminescent signal and determine the localization of the reporter. Bioluminescence tomography is an inverse problem that uses the optical signals on the surface of the animal body to quantitatively reconstruct the bioluminescent source distribution. This technique has attracted more and more attention recently [4,10,15]. To reconstruct the bioluminescent source distribution, the solution of a forward model is iteratively utilized to provide predicted measurement data, thus a good solver for the forward problem is required.

Light propagation in biological tissue is governed by the 3d steady-state RTE:

$$c\partial_x\Psi+s\partial_y\Psi+\zeta\partial_z\Psi+\sigma_T\Psi=\frac{1}{4\pi}(\sigma_T-\sigma_A)\int_{S^2}\mathcal{P}(\mathbf{u}',\mathbf{u})\Psi(x,y,z,\mathbf{u}')d\mathbf{u}'+q,\quad (x,y,z)\in\Omega, \tag{1.1}$$

with inflow boundary conditions

$$\Psi(x,y,z,\mathbf{u})=\tilde{\Psi}(x,y,z,\mathbf{u}),\qquad \text{for }\ \mathbf{u}\cdot\mathbf{n}<\mathbf{0},\ (x,y,z)\in\partial\Omega. \tag{1.2}$$

Here $\mathbf{u}=(c,s,\zeta)$ with $c^2+s^2+\zeta^2=1$ is a 3d vector on a unit sphere describing the direction of photons. Ω is a bounded domain with boundary $\partial\Omega$, while $\mathbf{n}$ is the outer normal vector of $\partial\Omega$. $\Psi(x,y,z,\mathbf{u})$ is the probability density of photons that move along the direction $\mathbf{u}$ at position (x,y,z). σ_T, σ_A are respectively the total cross section and absorption cross section. They depend on space but are independent of $\mathbf{u}$. $q=q(x,y,z,\mathbf{u})$ is the source term that can depend both on space and velocity. $\mathcal{P}(\mathbf{u},\mathbf{u}')$ gives the probability that photons change their direction from $\mathbf{u}'$ to $\mathbf{u}$ after scattering.

In this paper, we consider the Henyey-Greenstein scattering kernel, which is usually used in biological tissue for ray transportation. Let θ be the angle between $\mathbf{u}$ and $\mathbf{u}'$, the Henyey-Greenstein kernel writes

$$\mathcal{P}(\mathbf{u}',\mathbf{u})=G(\mathbf{u}\cdot\mathbf{u}')=G(\cos\theta)=\frac{1}{2}\frac{1-g^2}{(1+g^2-2g cos\theta)^{\frac{3}{2}}}, \tag{1.3}$$

where θ is the included angle between $\mathbf{u}$, $\mathbf{u}'$ and $0\le g<1$ is the asymmetry factor describing the anisotropy strength of the scattering kernel. $g=0$ is the isotropic scattering case and that g is close to 1 indicates the highly forward peaked case. The Henyey-Greenstein kernel satisfies

$$\int_{-1}^{1}G(cos\theta)d(\cos\theta)=1, \tag{1.4}$$

and

$$\int_{-1}^{1}cos\theta G(cos\theta)d(\cos\theta)=g. \tag{1.5}$$