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Nonlocal Wave Propagation in Unbounded Multi-Scale Media

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Abstract. This paper focuses on the simulation of nonlocal wave propagations in unbounded multi-scale mediums. To this end, we consider two issues: (a) the design of artificial/absorbing boundary conditions; and (b) the construction of an asymptotically compatible (AC) scheme for the nonlocal operator with general kernels. The design of ABCs facilitates us to reformulate unbounded domain problems into bounded domain problems. The construction of AC scheme facilitates us to simulate nonlocal wave propagations in multi-scale mediums. By applying the proposed ABCs and the proposed AC scheme, we investigate different wave propagation behaviors in the "local" and nonlocal mediums through numerical examples.

AMS subject classifications: 49M25, 65N22, 65N06, 65R20, 82C21, 46N40, 45A05

Key words: Asymptotically compatible, absorbing boundary method, nonlocal wave equation, nonlocal wave propagation, multi-scale mediums.

1 Introduction

Nonlocal models have attracted much attention during the last decade owing to its potentially promising applications in various disciplines of science and engineering, such as the peridynamical (PD) theory of continuum mechanics, the nonlocal theory of wave propagation, and the modeling of nonlocal diffusion process, see [4, 8, 22, 34, 39]. While most existing nonlocal models are formulated on bounded domains with volume constraints, there are indeed situations in which models in infinite domains are more reasonable, such as wave propagation in an exceedingly large sample.

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The PD model has been proposed for studying various processes including the dynamic fracture [7, 22]. An important feature of PD models is that it involves a horizon parameter, which characterizes the interaction range of the medium. When the horizon tends to zero, the PD models formally converge to local models, wherever the latter are well-defined. In a multi-scale medium, the horizon might vary significantly within a continuum of scales. In this situation, the PD model could potentially couple together the local and nonlocal models in different regions via heterogeneous localization [28].

In this paper, we consider the numerical computation of the following Cauchy problem of nonlocal wave equations

$$(\partial_t^2+\mathcal{L}_\gamma)q(x,t)=f(x,t), \quad x\in\mathbb{R}, \quad t>0, \tag{1.1}$$

$$q(x,0)=\psi_0(x), \quad \partial_t q(x,0)=\psi_1(x), \quad x\in\mathbb{R}, \tag{1.2}$$

where $q(x,t)$ represents the displacement field, $\psi_k(x)$ (k=0,1) are the initial values, $f(x,t)$ is the body force. The linear nonlocal operator $\mathcal{L}_\gamma$ (associated with γ) is defined as

$$\mathcal{L}_\gamma q(x)=\int_{\mathbb{R}}[q(x)-q(y)]\gamma\left(y-x,\frac{y+x}{2}\right)dy, \tag{1.3}$$

where the kernel function γ is nonnegative and satisfies

$$\gamma(-\alpha,\beta)=\gamma(\alpha,\beta), \quad \forall\alpha,\beta\in\mathbb{R}, \quad \text{and} \quad \gamma(\alpha,\beta)=0, \text{ if } |\alpha|>\delta>0. \tag{1.4}$$

Note that the horizon δ is allowed to be variable. If γ depends on the second variable, the nonlocal medium is spatially inhomogeneous.

The goal of this paper is to develop an efficient numerical scheme to compute the solution of problem (1.1)-(1.2) confined into a local region of physical interest. We are facing two difficulties:

- The simulations are implemented in multi-scale media. This necessitates us to develop numerical schemes which should be consistent to both its local limiting model ($\delta\to 0$) and the nonlocal model itself ($\delta=\mathcal{O}(1)$);
- The definition domain is unbounded. This necessitates the design of accurate artificial/absorbing boundary conditions (ABCs) to truncate the computational domain and minimize the fictitious wave reflection from the artificial boundary.

In terms of the first difficulty, it is known that the simulations in multi-scale media need to employ asymptotic compatibility (AC) schemes, a concept developed in [26, 27] to discretize the nonlocal operator. One may also refer to review papers [5, 6]. For a multi-scale model and its numerical simulation, the AC property is a key ingredient to ensure that numerical solutions of nonlocal models converge to the correct local limiting solution, as the mesh size tends to zero and the nonlocal effect diminishes. In this paper, we allow the kernel to be heterogeneous [28] and the following diffusion coefficient

$$0<\sigma(x)=\frac{1}{2}\int_{\mathbb{R}}s^2\gamma(s,x)ds<\infty, \tag{1.5}$$