

Fast Finite Element Method for the Three-Dimensional Poisson Equation in Infinite Domains

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Received 23 June 2017; Accepted (in revised version) 28 September 2017

Abstract. We aim at a fast finite element method for the Poisson equation in three-dimensional infinite domains. Both the exterior and strip-tail problems are considered. By introducing a suitable artificial boundary and imposing the exact boundary condition of Dirichlet-to-Neumann (DtN) type, we reduce the original infinite domain problem into a truncated finite domain problem. The point is how to efficiently implement this exact artificial boundary condition. The traditional modal expansion method is hard to apply for the strip-tail problem with a general cross section. We develop a fast algorithm based on the Padé approximation for the square root function involved in the exact artificial boundary condition. The most remarkable advantage of our method is that it is unnecessary to compute the full eigen system associated with the Laplace-Beltrami operator on the artificial boundary. Besides, compared with the modal expansion method, the computational cost of the DtN mapping is significantly reduced. We perform a complete numerical analysis on the fast algorithm. Some numerical examples are presented to demonstrate the effectiveness of the proposed method.

AMS subject classifications: 35A35, 65N12, 65N30

Key words: Infinite domain problems, exact artificial boundary conditions, fast algorithms.

1 Introduction

Partial differential equations in infinite domains arise in many application fields. Roughly, these problems can be categorized into two subgroups, exterior problems and strip-tail problems. Wave propagation in the free space and flow in a long pipe are typical examples. The infiniteness of the definition domains presents a practical numerical difficulty, since a direct discretization method would result in an algebraic system which involves an infinite number of degrees of freedom. This is unacceptable if no sophisticated solving technique can be developed.

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The integral equation method (IEM), and its coupling with finite element method, is a very classical method which is potentially capable of handling the linear infinite domain problems. The key ingredient of IEM is the ambient Green's function, with which the solution in infinite domains can be expressed as an appropriate integral involving the Green's function. There are a huge amount of works addressing on this method under different problem settings. Here, we do not attempt to make even a brief review. The interested readers are referred to [3,4] for more detailed discussions. However, we need to point out that the IEM is less compatible with the strip-tail problems, since for this kind of problems, the ambient Green's function is generally hard to formulate into a convenient closed form.

Artificial boundary method is another popular method which is capable of handling the infinite domain PDE problems. With regard to the Poisson equation, Han and his collaborators [8,9] derived exact boundary conditions based on the eigenmode expansion on the circular or spherical artificial boundaries. After truncating the Fourier series terms, a sequence of approximate artificial boundary conditions with controlled accuracy can then be derived. Another approach for solving the problem is to derive local boundary conditions, within which the Neumann data at a specific point on the artificial boundary was only related to the function and its derivative values. This method was proposed in [15–17]. For the discrete artificial boundary conditions that use numerical techniques to construct artificial boundary conditions, one can refer to [18]. Analogous idea was applied also in [10–12,14] for other kind of artificial boundaries. The eigenmode expansion method has two drawbacks. The first one is that for strip-tail problems with a general cross section, the analytical eigensystem is hard to derive, and one has to resort to some numerical eigensolver. This treatment turns out to be a subtle and complicated issue, especially for problems with a large number of degrees of freedom. The second one arises from the relatively large number of modes which are needed to maintain the optimal accuracy of interior discretization. As to the two-dimensional exterior problem of Poisson equation, the analysis presented in [8] reveals that the number of eigenmodes should be on the order of h^{-1} (h being the mesh size) when the artificial boundary is chosen as the tightest possible.

The Schwarz iteration method based on the domain decomposition technique can also be applied to solve the infinite domain problems. By introducing two concentric spherical surfaces, Savchenko et al. [13] developed an overlapping Schwarz iteration method of Dirichlet-to-Dirichlet type, to solve the three-dimensional exterior problem of Poisson equation. They applied the finite element discretization for the finite interior domain subproblem, and the analytical Poisson's integral formula for the infinite exterior domain subproblem. A convergence rate analysis was performed for the iterative process.

Some sophisticated techniques have been also developed to handle the infinite domain problems under some special settings. With regard to the Poisson equation, Lai et al. [5] introduced an auxiliary sphere to separate the infinite domain and applied the Kelvin's inversion to transform the exterior domain into a finite spherical domain. They employed finite difference method together with the discrete Fourier transform to com-