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Ground States and Energy Asymptotics of the Nonlinear Schrödinger Equation with a General Power Nonlinearity

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Abstract. We study analytically the existence and uniqueness of the ground state of the nonlinear Schrödinger equation (NLSE) with a general power nonlinearity described by the power index $\sigma \geq 0$. For the NLSE under a box or a harmonic potential, we can derive explicitly the approximations of the ground states and their corresponding energy and chemical potential in weak or strong interaction regimes with a fixed nonlinearity σ. Besides, we study the case where the nonlinearity $\sigma \to \infty$ with a fixed interaction strength. In particular, a bifurcation in the ground states is observed. Numerical results in 1D and 2D will be reported to support our asymptotic results.

AMS subject classifications: 35B40, 35P30, 35Q55, 65N25

Key words: Nonlinear Schrödinger equation, ground state, energy asymptotics, repulsive interaction.

1 Introduction

In this paper, we will consider the dimensionless time-independent nonlinear Schrödinger equation (NLSE) in d dimensions ($d=3,2,1$) [3,4,8,13,21,22]

$$\left[-\frac{1}{2}\Delta + V(\mathbf{x}) + \beta|\phi(\mathbf{x})|^{2\sigma}\right]\phi(\mathbf{x}) = \mu\phi(\mathbf{x}), \qquad \mathbf{x}\in\Omega\subseteq\mathbb{R}^d, \tag{1.1}$$

where $\phi := \phi(\mathbf{x})$ is the wave function (or eigenfunction) satisfying the normalization condition

$$\|\phi\|_2^2 := \int_\Omega |\phi(\mathbf{x})|^2 d\mathbf{x} = 1, \tag{1.2}$$

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$V:=V(\mathbf{x})$ is a given real-valued potential bounded below, $\beta\geq 0$ is a dimensionless constant describing the repulsive (defocussing) interaction strength, $\sigma\geq 0$ represents different nonlinearities, and the eigenvalue (or chemical potential in physics literature) $\mu:=\mu(\phi)$ is defined as [3,4,13,21]

$$\mu(\phi)=E(\phi)+\frac{\sigma\beta}{\sigma+1}\int_{\Omega}|\phi(\mathbf{x})|^{2\sigma+2}d\mathbf{x}, \tag{1.3}$$

with the energy $E:=E(\phi)$ defined as [4,22]

$$E(\phi)=\int_{\Omega}\left[\frac{1}{2}|\nabla\phi(\mathbf{x})|^2+V(\mathbf{x})|\phi(\mathbf{x})|^2+\frac{\beta}{\sigma+1}|\phi(\mathbf{x})|^{2\sigma+2}\right]d\mathbf{x}. \tag{1.4}$$

If Ω is bounded, the homogeneous Dirichlet BC, i.e. $\phi(\mathbf{x})|_{\partial\Omega}=0$, needs to be imposed. Thus, the time-independent NLSE (1.1) is a nonlinear eigenvalue problem under the constraint $\|\phi\|=1$. It is a mean field model arising from Bose-Einstein condensates (BECs) [2,3,13,16], nonlinear optics [12], and some other applications [1,21,22] that can be obtained from the N-body Schrödinger equation via the Hartree ansatz and mean field approximation [4,11,19,21]. When $\beta=0$ or $\sigma=0$, it collapses to the time-independent Schrödinger equation. When $\sigma=1$, the nonlinearity is cubic and it is usually known as the Gross-Pitaevskii equation (GPE) [3,13,14,21]. When $\sigma=2$, the nonlinearity is quintic and it is used to model the Tonks-Girardeau (TG) gas in BEC [15,17,19,23].

The ground state of the NLSE (1.1) is usually defined as the minimizer of the nonconvex minimization problem (or constrained minimization problem) [3,4,13,16]

$$\phi_g=\underset{\phi\in S}{\operatorname{argmin}}\,E(\phi), \tag{1.5}$$

where $S=\{\phi\,|\,\|\phi\|_2^2:=\int_\Omega|\phi(\mathbf{x})|^2d\mathbf{x}=1,\ E(\phi)<\infty,\ \phi|_{\partial\Omega}=0 \text{ if } \Omega \text{ is bounded}\}$. Since S is a nonconvex set, the problem (1.5) is a nonconvex minimization problem. It is easy to see that the ground state ϕ_g satisfies the time-independent NLSE (1.1). Hence it is an eigenfunction (or stationary state) of (1.1) with the least energy.

The main purpose of this paper is to study the existence and uniqueness of the ground state of the NLSE and its approximations under a box or a harmonic potential in special parameter regimes. The rest of this paper is organized as follows. In Section 2, we study analytically the existence, uniqueness and nonexistence of the ground state of the NLSE. In Section 3, we derive the ground state approximations and energy asymptotics under a harmonic potential for different β's and σ's. Similar results are presented in Section 4 for the NLSE under a box potential. Some conclusions are drawn in Section 5.

2 Existence and uniqueness

In this section, we will generalize the existence and uniqueness results for the GPE case [4,20,24], where $\sigma=1$, to a general case with a nonnegative σ. For simplicity, we introduce