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Hermite Spectral Collocation Methods for Fractional PDEs in Unbounded Domains

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Abstract. This work is concerned with spectral collocation methods for fractional PDEs in unbounded domains. The method consists of expanding the solution with proper global basis functions and imposing collocation conditions on the Gauss-Hermite points. In this work, two Hermite-type functions are employed to serve as basis functions. Our main task is to find corresponding differentiation matrices which are computed recursively. Two important issues relevant to condition numbers and scaling factors will be discussed. Applications of the spectral collocation methods to multi-term fractional PDEs are also presented. Several numerical examples are carried out to demonstrate the effectiveness of the proposed methods.

AMS subject classifications: 33C45, 34K37, 35R11, 65M70

Key words: Fractional PDEs, Hermite polynomials/functions, unbounded domain, spectral collocation methods.

1 Introduction

Many systems in science and engineering can be more accurately described by using fractional partial differential equations (PDEs) rather than the traditional approaches [1, 2, 24]. This leads to an intensive investigation over the past two decades on efficient numerical methods for fractional PDEs. Among others, the finite difference method and the finite element method are two widely investigated methods in this direction, see, e.g., [11, 13, 14, 18, 27, 32, 35] and references therein.

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Another powerful approach for fractional models is the spectral methods. In this approach, the key is to construct suitable basis functions to handle to the solution singularities. Along this direction, a recent advance is brought by Karniadakis and co-authors who proposed the so-called Jacobi *poly-fractonomials* based spectral methods [36]. These bases are eigenfunctions of the corresponding fractional and tempered fractional Sturm-Liouville problems. Another approach that employs the generalized Jacobi functions (GJFs) is proposed by Shen et al. [6]. Those bases are adapted to the fractional operator, as a fractional derivative of poly-fractonomials/GJFs is simply another poly-fractonomials/GJFs with different parameter. Consequently, fractional derivatives become a local operator in the physical space spanned by poly-fractonomials/GJFs, and this property leads to very efficient spectral methods for fractional PDEs in bounded domains. The poly-fractonomials/GJFs have been successfully applied to various fractional models [5, 16, 21, 30]. However, compared to fractional PDEs in bounded domain, little works have been done for fractional PDEs defined on unbounded domains. Very recently, a spectral method for fraction differential equations in the half line is proposed in [17, 21] — using the generalized Largurre functions as bases — extending the idea of [36].

When this paper is prepared, we noticed very recent work of Mao and Shen [23] who proposed both the spectral Galerkin and collocation method for fractional PDEs in unbounded domains. However, the collocation method therein relies on an equivalent formulation in frequency space by the Fourier transform, and performs collocation methods to the equivalent formulation that involve forward/backward Hermite transform. In contrast, our collocation methods are *direct* methods that based on the derivation of explicit DMs. Moreover, our approach can be easily applied to nonlinear problems as the differentiation matrices are constructed explicitly.

In this work, we aim at designing spectral collocation methods for fractional PDEs in unbounded domains (the whole space $\mathbb{R}^d$). To better demonstrate our idea, we consider the following model equation:

$$\begin{cases} (-\Delta)^{\alpha/2}u(\mathbf{x})+\gamma f(u)=g(\mathbf{x}), & \mathbf{x}\in\mathbb{R}^d, \\ u(\mathbf{x})=0, & |\mathbf{x}|\to\infty, \end{cases} \tag{1.1}$$

where $f(u)$ is a linear/nonlinear function of u, and the fractional Laplace operator is defined as [19]

$$(-\Delta)^{\alpha/2}u(\mathbf{x})=C_{n,\alpha}\int_{\mathbb{R}^d}\frac{u(\mathbf{x})-u(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{n+\alpha}}d\mathbf{y}, \quad \text{with} \quad C_{n,\alpha}=\frac{\alpha 2^{\alpha-1}\Gamma\left(\dfrac{\alpha+n}{2}\right)}{\pi^{n/2}\Gamma\left(\dfrac{2-\alpha}{2}\right)}. \tag{1.2}$$

Notice that the fractional Laplace operator $(-\Delta)^{\alpha/2}$, where $0<\alpha<2$, recovers the standard Laplace operator as $\alpha\to 2$.

For such problems that are defined on the whole space $\mathbb{R}^n$, there are alternative (equivalent) ways to define the fractional Laplace operator. For example, it can be de-