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A Cartesian Grid-Based Boundary Integral Method for an Elliptic Interface Problem on Closely Packed Cells

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Abstract. In this work, we propose a second-order version and a fourth-order version of a Cartesian grid-based boundary integral method for an interface problem of the Laplace equation on closely packed cells. When the cells are closely packed, the boundary integrals involved in the boundary integral formulation for the interface problem become nearly singular. Direct evaluation of the boundary integrals has accuracy issues. The grid-based method evaluates a boundary integral by first solving an equivalent, simple interface problem on a Cartesian grid with a fast Fourier transform based Poisson solver, then interpolating the grid solution to get values of the boundary integral at discretization points of the interface. The grid-based method presents itself as an alternative but accurate numerical method for evaluating nearly singular, singular and hyper-singular boundary integrals. This work can be regarded as a further development of the kernel-free boundary integral method [W.-J. Ying and C. S. Henriquez, A kernel-free boundary integral method for elliptic boundary value problems, Journal of Computational Physics, Vol. 227 (2007), pp. 1046-1074] for problems in unbounded domains. Numerical examples with both second-order and fourth-order versions of the grid-based method are presented to demonstrate accuracy of the method.

AMS subject classifications: 35J05, 65N06, 65N38

Key words: Laplace equation, cell suspension, inclusion of grains, boundary integral method, Cartesian grid method, finite difference method, fast Fourier transform.

1 Introduction

Let $\Omega_i \subset \mathbb{R}^2$ be a bounded open set with smooth boundary, which may have multiple disconnected components, $\Omega_e = \mathbb{R}^2 \backslash \bar{\Omega}_i$ be the unbounded, complementary domain and Γ be the interface, the common boundary of Ω_i and Ω_e. When the interface Γ has multiple

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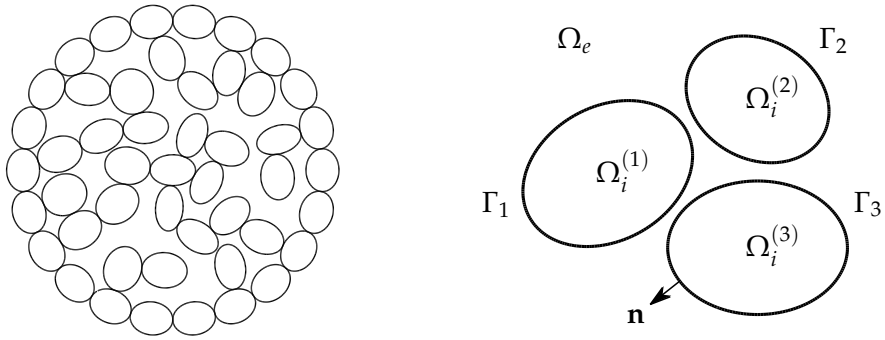

Figure 1: Multi-component interfaces: (a) closely packed cells, (b) illustration of symbols for the interface and subdomains (cells).

components, we write $\Gamma=\bigcup_{k=1}^{K}\Gamma_k$, and assume each component Γ_k is a simple closed curve. We call the subdomain enclosed by each interface component Γ_k a cell, denoted by $\Omega_i^{(k)}$.

Let $\mathbf{p}=(x,y)\in\mathbb{R}^2$ be a point in space. Suppose $\Phi_i(\mathbf{p})$ and $\Phi_e(\mathbf{p})$ are two unknown potential functions on Ω_i and Ω_e, respectively. They satisfy the Laplace equation

$$\triangle\Phi_i(\mathbf{p})=0,\quad \text{in } \Omega_i \tag{1.1}$$

and

$$\triangle\Phi_e(\mathbf{p})=0,\quad \text{in } \Omega_e. \tag{1.2}$$

Let

$$\Phi(\mathbf{p})=\begin{cases}\Phi_i(\mathbf{p}), & \mathbf{p}\in\Omega_i,\\ \Phi_e(\mathbf{p}), & \mathbf{p}\in\Omega_e.\end{cases}$$

In general, the function $\Phi(\mathbf{p})$ is discontinuous across the interface Γ. Let

$$\Phi_i(\mathbf{p})-\Phi_e(\mathbf{p})=V_m(\mathbf{p})\quad \text{on } \Gamma, \tag{1.3}$$

where $V_m(\mathbf{p})$ will be known. Assume the conductivities σ_i and σ_e on Ω_i and Ω_e are constant but distinct ($\sigma_i\neq\sigma_e$). Let

$$\sigma_i\frac{\partial\Phi_i(\mathbf{p})}{\partial\mathbf{n}_\mathbf{p}}-\sigma_e\frac{\partial\Phi_e(\mathbf{p})}{\partial\mathbf{n}_\mathbf{p}}=J_m(\mathbf{p})\quad \text{on } \Gamma. \tag{1.4}$$

Here, $\mathbf{n}_\mathbf{p}$ is the unit normal vector pointing from the bounded domain Ω_i to the unbounded domain Ω_e at point $\mathbf{p}\in\Gamma$; $J_m(\mathbf{p})$ will be known, too. We assume the potential function $\Phi_e(\mathbf{p})$ satisfies the far field condition

$$\Phi_e(\mathbf{p})\to 0,\quad \text{as } |\mathbf{p}|=\sqrt{x^2+y^2}\to\infty. \tag{1.5}$$