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Truncated Gaussian RBF Differences are Always Inferior to Finite Differences of the Same Stencil Width

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Abstract. Radial basis functions (RBFs) can be used to approximate derivatives and solve differential equations in several ways. Here, we compare one important scheme to ordinary finite differences by a mixture of numerical experiments and theoretical Fourier analysis, that is, by deriving and discussing analytical formulas for the error in differentiating $\exp(ikx)$ for arbitrary k.'Truncated RBF differences" are derived from the same strategy as Fourier and Chebyshev pseudospectral methods: Differentiation of the Fourier, Chebyshev or RBF interpolant generates a differentiation matrix that maps the grid point values or samples of a function $u(x)$ into the values of its derivative on the grid. For Fourier and Chebyshev interpolants, the action of the differentiation matrix can be computed indirectly but efficiently by the Fast Fourier Transform (FFT). For RBF functions, alas, the FFT is inapplicable and direct use of the dense differentiation matrix on a grid of N points is prohibitively expensive ($\mathcal{O}(N^2)$) unless N is tiny. However, for Gaussian RBFs, which are exponentially localized, there is another option, which is to *truncate* the dense matrix to a banded matrix, yielding "truncated RBF differences". The resulting formulas are identical in form to finite differences except for the difference weights. On a grid of spacing h with the RBF as $\phi(x)=\exp(-\alpha^2(x/h)^2)$,

$$\frac{df}{dx}(0)\approx\sum_{m=1}^{\infty} w_m\{f(mh)-f(-mh)\},$$

where without approximation $w_m=(-1)^{m+1}2\alpha^2/\sinh(m\alpha^2)$. We derive explicit formula for the differentiation of the linear function, $f(X)\equiv X$, and the errors therein. We show that Gaussian radial basis functions (GARBF), when truncated to give differentiation formulas of stencil width $(2M+1)$, are significantly less accurate than $(2M)$-th order finite differences of the same stencil width. The error of the infinite series ($M=\infty$) decreases exponentially as $\alpha\rightarrow 0$. However, truncated GARBF series have a second error (truncation error) that grows exponentially as $\alpha\rightarrow 0$. Even for $\alpha\sim\mathcal{O}(1)$

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where the sum of these two errors is minimized, it is shown that the finite difference formulas are always superior. We explain, less rigorously, why these arguments extend to more general species of RBFs and to an irregular grid. There are, however, a variety of alternative differentiation strategies which will be analyzed in future work, so it is far too soon to dismiss RBFs as a tool for solving differential equations.

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1 Introduction

Radial basis functions (RBFs) are a popular method for multidimensional interpolation on irregular or scattered grids [7, 8, 35, 38]. RBFs are now widely applied for solving differential equations in many different branches of physics and engineering [1, 9, 11–15, 20, 22–27, 29, 31, 34, 36, 37, 39]. The form of the approximation is very simple: in any number of dimensions d,

$$f(\vec{x}) \approx \sum_{j=1}^{N} \lambda_j \phi\left(||\vec{x}-\vec{c}_j||_2\right) \qquad \vec{x} \in R^d \tag{1.1}$$

for some function $\phi(r)$ and some set of N points $\vec{c}_j$, which are called the "centers". The coefficients λ_j are usually found by interpolation at a set of points $\vec{x}_k$ that may or may not coincide with the centers. Under fairly mild conditions on ϕ, the interpolation problem is provably solvable even when the interpolation points and centers are scattered randomly over an irregularly-shaped domain.

To solve partial differential equations by RBFs, it is obviously necessary to have a strategy for differentiating the RBF series and evaluating the derivative sums at the grid points. There are at least seven distinct strategies for RBF differentiation. Curiously, there hasn't been a careful review comparing these different options; for the sake of brevity none is offered here. We shall only assert, as will be demonstrated in our future work, that all RBF differentiation methods have liabilities. The most common strategy is to apply RBFs as a global pseudospectral method which generates a *dense* matrix which is very expensive to manipulate; consequently, most applications even in multiple space dimensions have used a thousand basis functions or less. Our modest goal is to perform a detailed analysis of the merits and failings of one particular strategy, dubbed "truncated RBF differences", which is much less expensive than the global, dense matrix approach.

Although many types of $\phi(r)$ have been used in the literature as reviewed in [17], we prefer to forgo a catalogue of limited results in favor of an in-depth examination of a single important case: that of Gaussian RBFs for which

$$\phi(x) \equiv \exp(-\epsilon^2 x^2). \tag{1.2}$$