COMMUNICATIONS IN COMPUTATIONAL PHYSICS
Vol. **5**, No. 1, pp. 84-107

Commun. Comput. Phys.
January 2009

A Stable Finite Difference Method for the Elastic Wave Equation on Complex Geometries with Free Surfaces

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Received 3 January 2008; Accepted (in revised version) 6 May 2008

Available online 15 July 2008

Abstract. A stable and explicit second order accurate finite difference method for the elastic wave equation in curvilinear coordinates is presented. The discretization of the spatial operators in the method is shown to be self-adjoint for free-surface, Dirichlet and periodic boundary conditions. The fully discrete version of the method conserves a discrete energy to machine precision.

AMS subject classifications: 65M06, 74B05, 86A15

Key words: Elastic wave equation, curvilinear grids, finite differences, stability, energy estimate, seismic wave propagation.

1 Introduction

The isotropic elastic wave equation governs the propagation of seismic waves caused by earthquakes and other seismic events. It also governs the propagation of waves in solid material structures and devices, such as gas pipes, wave guides, railroad rails and disc brakes. In the vast majority of wave propagation problems arising in seismology and solid mechanics there are free surfaces, i.e. boundaries with vanishing normal stresses. These free surfaces have, in general, complicated shapes and are rarely flat.

Another feature, characterizing problems arising in these areas, is the strong heterogeneity of the media, in which the problems are posed. For example, on the characteristic length scales of seismological problems, the geological structures of the earth can be described by piecewise smooth functions with jump discontinuities. However, compared to the wavelengths, which can be resolved in computations, the material properties vary

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rapidly. Large spatial contrasts are also found in solid mechanics devices composed of different materials welded together.

The presence of curved free surfaces, together with the typical strong material heterogeneity, makes the design of stable, efficient and accurate numerical methods for the elastic wave equation challenging. Today, many different classes of numerical methods are used for the simulation of elastic waves. Early on, most of the methods were based on finite difference approximations of space and time derivatives of the equations in second order differential form (displacement formulation), see for example [1,2]. The main problem with these early discretizations were their inability to approximate free surface boundary conditions in a stable and fully explicit manner, see, e.g., [10, 11, 18, 21]. The instabilities of these early methods were especially bad for problems with materials with high ratios between the P-wave (C_p) and S-wave (C_s) velocities.

For rectangular domains, a stable and explicit discretization of the free surface boundary conditions is presented in the paper [17] by Nilsson et al. In summary, they introduce a discretization that use boundary-modified difference operators for the mixed derivatives in the governing equations. Nilsson et al. show that the method is second order accurate for problems with smoothly varying material properties and stable under standard CFL constraints, for *arbitrarily* varying material properties.

In this paper we generalize the results of Nilsson et al. to curvilinear coordinate systems, allowing for simulations on non-rectangular domains. Using summation by parts techniques, we show how to construct a corresponding stable discretization of the free surface boundary condition on curvilinear grids. We also prove that the discretization is stable and energy conserving both in semi-discrete and fully discrete form. As for the Cartesian method in [17], the stability and conservation results holds for *arbitrarily varying material properties*. By numerical experiments it is established that the method is second order accurate.

The strengths of the proposed method are its ease of implementation, its (relative to low order unstructured grid methods) efficiency, its geometric flexibility, and, most importantly, its "bullet-proof" stability. The proposed method is second order accurate for materials with smoothly varying properties. However, it has been known for a long time [14] that second order methods are less efficient than higher (4th or more) order methods. When the material properties are only piecewise smooth (as e.g. in seismology), the difference in efficiency between high and low order accurate methods is not as pronounced, see, e.g., [4,9]. For such problems the formal order of accuracy (for both high and low order methods) is reduced to one, but as has been shown in [4], the higher order methods produce more accurate results. Although we believe that the present method is reasonably competitive for strongly heterogeneous materials, it would be of great interest to derive a similarly "bullet-proof" fourth or higher order accurate method.

There are of course many other numerical methods capable of handling general geometries. Two recent finite difference methods are the traction image method for curvilinear grids [22], and the embedded boundary method by Lombard et al. described in [15]. Both these methods use dissipative time-integration schemes while our method is non-