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Efficient Preconditioners for a Shock Capturing Space-Time Discontinuous Galerkin Method for Systems of Conservation Laws

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Abstract. An entropy stable fully discrete shock capturing space-time Discontinuous Galerkin (DG) method was proposed in a recent paper [20] to approximate hyperbolic systems of conservation laws. This numerical scheme involves the solution of a very large nonlinear system of algebraic equations, by a Newton-Krylov method, at every time step. In this paper, we design efficient preconditioners for the large, non-symmetric linear system, that needs to be solved at every Newton step. Two sets of preconditioners, one of the block Jacobi and another of the block Gauss-Seidel type are designed. Fourier analysis of the preconditioners reveals their robustness and a large number of numerical experiments are presented to illustrate the gain in efficiency that results from preconditioning. The resulting method is employed to compute approximate solutions of the compressible Euler equations, even for very high CFL numbers.

AMS subject classifications: 65M60, 35L65

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1 Introduction

Hyperbolic systems of conservation laws are systems of nonlinear partial differential equations that model many interesting phenomena in physics and engineering. Examples include the shallow water equations of oceanography, the compressible Euler equations of aerodynamics, the magnetohydrodynamics (MHD) equations of plasma physics and the equations of nonlinear elastodynamics [6].

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It is well known that solutions of systems of conservation laws can form discontinuities such as *shock waves*, even when the initial data is smooth. Hence, the solutions of systems of conservation laws is interpreted in the *weak* (distributional) sense. These weak solutions are not necessarily unique. Further admissibility criteria in the form of *entropy* conditions need to be imposed in order to guarantee uniqueness. Detailed formulation of entropy solutions is provided in Section 2 and in standard textbooks such as [6]. But in fact, recent work [11, 14] suggests that the notion of solutions has to be further weakened into the more general *entropy measure valued solutions* in order to obtain wellposedness for multi-dimensional systems of conservation laws. Measure valued solutions are space-time parametrized probability measures and are shown to be natural limits of numerical approximations [11].

1.1 Numerical schemes

Given the nonlinear nature of systems of conservation laws, it is not possible to obtain explicit solution formulas. Consequently, numerical methods play a key role in the study of these equations. Various numerical methods of the finite difference, finite volume, finite element and spectral type are available for the approximation of systems of conservation laws. In particular, the finite volume (difference) methods, that update cell averages (point values) in terms of numerical fluxes, are heavily used [28]. The numerical fluxes are obtained by using exact or approximate Riemann solvers. Higher-order spatial accuracy results from non-oscillatory piecewise polynomial reconstruction in each cell. Reconstruction procedures such as TVD [28], ENO [17] and WENO [30] are typically employed. Higher order temporal accuracy is achieved by using strong stability preserving (SSP) Runge-Kutta (RK) time integrators. An alternative to high-order finite volume methods is the discontinuous Galerkin finite element method [4, 5]. At lowest (first) order, these methods reduce to the standard finite volume method. However, high-order accuracy is obtained by using piecewise polynomial test and trial functions in each element. Limiters are employed to damp oscillations near shocks. Temporal accuracy is again increased by using SSP-RK methods. High-order finite volume methods and RKDG methods have been very successful in carrying out realistic large scale simulations of conservation laws [31].

The key questions in the numerical analysis of systems of conservation laws are that of *stability* and *convergence* of numerical schemes [15]. These questions have been carefully investigated in the simple case of scalar conservation laws. For this class of problems, first-order monotone schemes [15] satisfy a discrete maximum principle, a discrete form of the entropy inequality as well as the TVD property. Hence, they can be shown to converge to the entropy solution. Similar results have also been derived for higher order schemes, see [12] and references therein. However, the questions of stability and convergence are much harder to tackle for systems of conservation laws. For such equations, *entropy stability*, i.e., compliance with a discrete form of the entropy inequality, seems to be a reasonable stability requirement for numerical schemes [32]. Entropy stable finite