Commun. Comput. Phys.
doi: 10.4208/cicp.scpde14.26s

Vol. **19**, No. 5, pp. 1191-1220
May 2016

Analyses and Applications of the Second-Order Cross Correlation in the Passive Imaging

Lingdi Wang[1], Wenbin Chen[2,*] and Jin Cheng[3]

[1] *School of Mathematical Sciences, Fudan University, Shanghai 200433, P.R. China.*
[2] *Shanghai Key Laboratory for Contemporary Applied Mathematics, School of Mathematical Sciences, Fudan University, Shanghai 200433, P.R. China.*
[3] *Key Laboratory for Information Science of Electromagnetic Waves and School of Mathematical Sciences, Fudan University, Shanghai 200433, P.R. China.*

Received 27 February 2015; Accepted (in revised version) 9 September 2015

Abstract. The first-order cross correlation and corresponding applications in the passive imaging are deeply studied by Garnier and Papanicolaou in their pioneer works. In this paper, the results of the first-order cross correlation are generalized to the second-order cross correlation. The second-order cross correlation is proven to be a statistically stable quantity, with respective to the random ambient noise sources. Specially, with proper time scales, the stochastic fluctuation for the second-order cross correlation converges much faster than the first-order one. Indeed, the convergent rate is of order $\mathcal{O}(T^{-1+\alpha})$, with $0<\alpha<1$. Besides, by using the stationary phase method in both homogeneous and scattering medium, similar behaviors of the singular components for the second-order cross correlation are obtained. Finally, two imaging methods are proposed to search for a target point reflector: One method is based on the imaging function, and has a better signal-to-noise rate; Another method is based on the geometric property, and can improve the bad range resolution of the imaging results.

AMS subject classifications: 35L10, 35R30, 35R60, 78A46

Key words: Second-order cross correlation, ambient noise sources, stationary phase method, passive imaging.

1 Introduction

It is well known that the full Green's function between two passive sensors could be estimated by cross correlating wave signals emitted by ambient noise sources and recorded

*Corresponding author. *Email addresses:* `10110180010@fudan.edu.cn` (L. Wang), `wbchen@fudan.edu.cn` (W. Chen), `jcheng@fudan.edu.cn` (J. Cheng)

by the corresponding sensors [26, 28, 29]. The ambient noise is generated by sources that are randomly distributed in space and are statistically stationary in time. The relation between the cross correlation and Green's function can be proven using the Helmholtz-Kirchhoff identity when the noise sources surround the observation region [3, 5, 24].

In geophysics, the ambient seismic noise is used to extract the Green's function, i.e., cross-correlating the seismic noise recorded by two stations to get the Green's function between these two stations, thus to improve the knowledge about subsurface structures, see [20] and references therein. This idea has been successfully applied to background velocity estimation and travel time estimation. For background velocity estimation, the first application was carried out in seismology to obtain an estimation of the background surface wave velocity map of a large part of the Earth, where the seismic stations were used as the passive senors, and the noise sources came from the nonlinear interaction of the ocean swell with the coast that generates surface waves [25]. It has also been applied to background velocity estimation from regional to local scales [19, 23], volcano monitoring [10, 11], and petroleum prospecting [12]. For travel time estimation, it was firstly proposed in helioseismology and seismology [13, 21, 22], where the cross correlation was used to retrieve the travel time information. Besides, the cross correlation has also been used in the noise source localization [1].

Another important application of the cross correlation is passive imaging, where only passive sensors are used to search for a target reflector buried in the homogeneous or scattering medium [4]. Specially, in Garnier's and Papanicolaou's pioneer works [14–16], the first-order cross correlation and corresponding applications in the passive imaging were deeply studied, and a self-consistent theoretical framework for the analysis of first-order cross correlation based interferometric imaging techniques was introduced, which was an extension of the coherent interferometric (CINT) imaging techniques [7, 8]. And the cross correlation has also been applied to the imaging problems in the random waveguide [2].

In this paper, based on Garnier's and Papanicolaou's framework [14–16], we consider the analyses and applications of the second-order cross correlation. The idea of the second-order cross correlation firstly appeared in Garnier's and Papanicolaou's work [14]. However, few theoretical analyses have been made, and there are no applications of the second-order cross correlation in the passive imaging. The main objective of our work is to present detailed and systemic analyses on the second-order cross correlation, and to apply it to the passive imaging. The rest of the paper is organized as follows. In Section 2, the results of the first-order cross correlation are recalled. Then, based on the first-order cross correlation, the second-order cross correlation is introduced in Section 3. Specially, we prove that the second-order cross correlation is a statistically stable quantity. In Section 4 and Section 5, with the stationary phase method, we analyze the behaviors of the second-order cross correlation in both homogeneous and scattering medium. Finally, two imaging methods are proposed in Section 6. One method is based on the imaging function, which is a direct generalization of the imaging method for the first-order case. Another method is based on the geometric property, which is original in this paper.

2 The results of the first-order cross correlation

The first-order cross correlation and corresponding applications in the passive imaging have been deeply studied in the pioneer works of Garnier's and Papanicolaou's [14–16]. In this section, we give a brief introduction. Let us consider the following wave equation in an infinite $\mathbb{R}^3$ domain,

$$\frac{1}{c^2(x)}\frac{\partial^2 u}{\partial t^2}-\Delta_x u=n^\varepsilon(t,x), \tag{2.1}$$

where the coefficient $c(x)$ is the wave speed which characterizes the property of the background medium. The source term n^ε models a random distribution of noise sources. In detail, it is assumed to be a zero-mean stationary Gaussian process with auto correlation function,

$$\langle n^\varepsilon(t_1,y_1)n^\varepsilon(t_2,y_2)\rangle=F^\varepsilon(t_2-t_1)\Gamma(y_1,y_2). \tag{2.2}$$

Here $\langle\cdot\rangle$ stands for the statistical average with respect to the distribution of the random sources. Function F^ε describes the behavior of the random sources in the temporal aspect. In physical, the decoherence time of the noise sources is assumed to be much smaller than typical travel times between sensors. Thus, we have the scaling relation as follows,

$$F^\varepsilon(t_2-t_1)=F\left(\frac{t_2-t_1}{\varepsilon}\right), \tag{2.3}$$

where $0<\varepsilon\ll 1$ physically stands for the ratio of these two time scales. Further, we assume that the Fourier transform $\hat{F}^\varepsilon$ is a nonnegative, even real-valued function. Moreover, $\hat{F}^\varepsilon$ is assumed to be compact supported and smooth enough. With Fourier transform, it is obvious that

$$\hat{F}^\varepsilon(\omega)=\varepsilon\hat{F}(\varepsilon\omega). \tag{2.4}$$

Here the Fourier transform is defined by

$$\hat{F}(\omega)=\int F(t)e^{\mathrm{i}\omega t}dt. \tag{2.5}$$

And the corresponding inverse transform is defined by

$$F(t)=\frac{1}{2\pi}\int \hat{F}(\omega)e^{-\mathrm{i}\omega t}d\omega. \tag{2.6}$$

On the other hand, function Γ describes the behavior of the random sources in the spacial aspect. It is the kernel of a symmetric nonnegative definite operator. For simplicity, n^ε here is assumed to be delta-correlated in space. Thus,

$$\Gamma(y_1,y_2)=\theta(y_1)\delta(y_1-y_2), \tag{2.7}$$

where the function θ characterizes the spatial support of the sources.

Denote $u(t,x)$ as the solution to (2.1), which can be expressed by

$$u(t,x)=\int G(s,x,y)n^{\varepsilon}(t-s,y)dsdy, \tag{2.8}$$

where $G(t,x,y)$ is the time-dependent Green's function in $\mathbb{R}^3$, i.e., the fundamental solution of the following wave equation,

$$\frac{1}{c^2(x)}\frac{\partial^2 G}{\partial t^2}-\Delta_x G=\delta(t)\delta(x-y). \tag{2.9}$$

Now given the wave signals recorded at x_1,x_2 for a total observation time T, the first-order cross correlation is defined by

$$C_T(\tau,x_1,x_2)=\frac{1}{T}\int_0^T u(t,x_1)u(t+\tau,x_2)dt. \tag{2.10}$$

Due to the random sources, C_T defined above is also stochastic. But fortunately, Garnier and Papanicolaou have proven that it is a statistically stable quantity, in the sense that for a large observation time T [14].

Theorem 2.1. *The expectation of C_T is independent of T, i.e.,*

$$\langle C_T(\tau,x_1,x_2)\rangle=C^{(1)}(\tau,x_1,x_2), \tag{2.11}$$

where $C^{(1)}$ is given by

$$C^{(1)}(\tau,x_1,x_2)=\int dy\int ds_1ds_2G(s_1,x_1,y)G(\tau+s_1+s_2,x_2,y)F^{\varepsilon}(s_2)\theta(y), \tag{2.12}$$

or equivalently by

$$C^{(1)}(\tau,x_1,x_2)=\frac{1}{2\pi}\int dy\int d\omega\overline{\hat{G}}(\omega,x_1,y)\hat{G}(\omega,x_2,y)\hat{F}^{\varepsilon}(\omega)e^{-\mathrm{i}\omega\tau}\theta(y). \tag{2.13}$$

Moreover, the first-order cross correlation C_T is a self-averaging quantity, i.e., as $T\to\infty$,

$$C_T(\tau,x_1,x_2)\to C^{(1)}(\tau,x_1,x_2), \tag{2.14}$$

in probability with respect to the distribution of the sources. More precisely, the fluctuations of C_T around its mean value $C^{(1)}$ are of order $\mathcal{O}(T^{-1/2})$.

Besides, in [14], Garnier and Papanicolaou also proved that the first-order cross correlation contains the information of travel time, which implies that the first-order cross correlation could be used in the travel time estimation.

Theorem 2.2. *As ε tends to zero, the first-order cross correlation $C^{(1)}(\tau,x_1,x_2)$ has either one or two singular components at $\tau=\pm\tau(x_1,x_2)$ if and only if the ray going through x_1 and x_2 reaches into the source region, that is, into the support of the function θ. Here $\tau(x_1,x_2)$ stands for the travel time of waves between x_1,x_2.*

Meanwhile, when there is a point reflector, Garnier and Papanicolaou proved that the first-order cross correlation contains the location information of the reflector, which is critical in the applications of passive imaging.

Theorem 2.3. *Assume that there is a point reflector at* z_r*, with reflectivity* $0<\sigma_r \ll 1$*. As* ε *tends to zero, the first-order cross correlation has two types of singular components, of order* $\mathcal{O}(1)$ *and* $\mathcal{O}(\sigma_r)$*, respectively. For the singular component of order* $\mathcal{O}(1)$*, it behaves the same as that described in Theorem* 2.2*. For the singular component of order* $\mathcal{O}(\sigma)$*,*

1. *If the ray going through* x_1 *and* z_r *extends into the source region and if* x_1 *is between* z_r *and the sources, then there is a singular component at* $\tau=\tau(x_1,z_r)+\tau(x_2,z_r)$*.*
2. *If the ray going through* x_1 *and* z_r *extends into the source region and if* z_r *is between* x_1 *and the sources, then there is a singular component at* $\tau=\tau(x_2,z_r)-\tau(x_1,z_r)$*.*
3. *If the ray going through* x_2 *and* z_r *extends into the source region and if* x_2 *is between* z_r *and the sources, then there is a singular component at* $\tau=-\tau(x_1,z_r)-\tau(x_2,z_r)$*.*
4. *If the ray going through* x_2 *and* z_r *extends into the source region and if* z_r *is between* x_2 *and the sources, then there is a singular component at* $\tau=\tau(x_2,z_r)-\tau(x_1,z_r)$*.*

In Theorem 2.3, we have four cases with different space configurations. Here and in what after, for case 1 and case 3, we call the corresponding space configuration as a daylight configuration, which means that the sensors are between the sources and the reflector. While for case 2 and case 4, we call the corresponding space configuration as a backlight configuration, which means that the reflector is between the sources and the sensors. Fig. 1 gives an illustration of both daylight and backlight configurations.

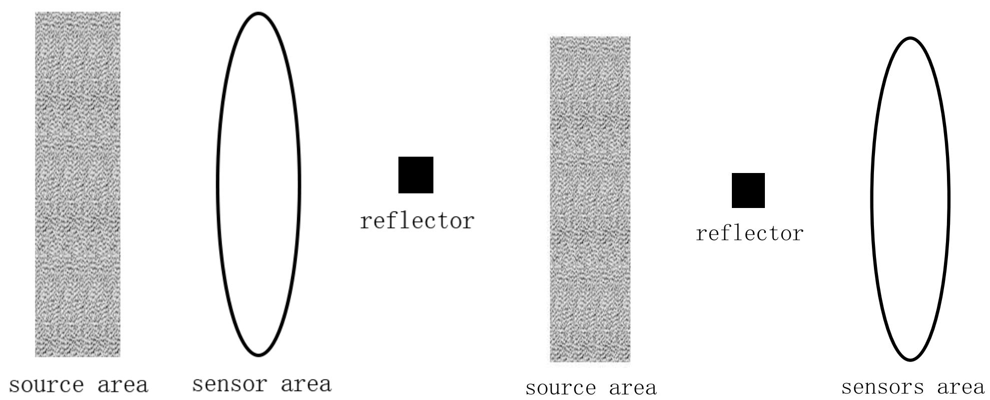

Figure 1: The illustrations of two space configurations. The sources are supported in the source area and the sensors are set in the sensor area. The left one corresponds to a daylight configuration, and the right one corresponds to a backlight configuration.

3 Second-order cross correlation

In this section, based on the definition of the first-order cross correlation, we introduce the second-order cross correlation.

Given any two sensors x_1,x_2, the first-order cross correlation has been defined by (2.10). Now if an auxiliary sensor x_3 is added, the second-order cross correlation can be defined as follows,

$$C_{T,T'}(\tau,x_1,x_2)=\int_{-T'}^{T'} C_T(\tau',x_3,x_1)C_T(\tau'+\tau,x_3,x_2)d\tau', \tag{3.1}$$

where C_T is given by (2.10), which is the first-order cross correlation.

Remark 3.1. The second-order cross correlation defined by (3.1) actually depends on x_3. However, if more auxiliary sensors could be introduced, the influence of x_3 could be eliminated by integration. In detail, the second-order cross correlation could be defined by

$$C_{T,T'}(\tau,x_1,x_2)=\int dx_3 \int_{-T'}^{T'} C_T(\tau',x_3,x_1)C_T(\tau'+\tau,x_3,x_2)d\tau', \tag{3.2}$$

which no longer depends on x_3.

Similarly, following Garnier's and Papanicolaou's results for the first-order cross correlation, we can prove that the second-order cross correlation, as a stochastic quantity, is also stable when T, T' are large enough.

Here and in what after, for convenience, we introduce the notation $\lesssim$ for 'less than or equal to' up to a constant.

Theorem 3.1. *Given positive constant $\alpha<1$, assume the two time scales satisfy $T'\lesssim T^{\alpha}$. Then as $T,T'\to\infty$, the second-order cross correlation satisfies,*

$$C_{T,T'}(\tau,x_1,x_2)\to C^{(2)}(\tau,x_1,x_2), \tag{3.3}$$

where

$$\begin{aligned} C^{(2)}(\tau,x_1,x_2)=&\frac{1}{2\pi}\int dy_1dy_2\theta(y_1)\theta(y_2)\int d\omega (F^{\varepsilon}(\omega))^2 e^{-\mathrm{i}\omega\tau} \\ &\times \hat{G}(\omega,x_3,y_1)\overline{\hat{G}}(\omega,x_1,y_1)\overline{\hat{G}}(\omega,x_3,y_2)\hat{G}(\omega,x_2,y_2). \end{aligned} \tag{3.4}$$

More precisely, the convergent rate is of order $\mathcal{O}(T^{-1+\alpha})$.

Proof. We will prove this theorem in two steps. Firstly, we will show that the expectation of $C_{T,T'}$ converges to $C^{(2)}$. Since $\hat{F}^{\varepsilon}$ is compact supported and smooth enough, by the property of Fourier transform, we have,

$$\begin{aligned} &\int_{-T'}^{T'} C^{(1)}(\tau',x_3,x_1)C^{(1)}(\tau'+\tau,x_3,x_2)d\tau' \to \int_{-\infty}^{\infty} C^{(1)}(\tau',x_3,x_1)C^{(1)}(\tau'+\tau,x_3,x_2)d\tau' \\ =&\frac{1}{2\pi}\int \overline{\hat{C^{(1)}}}(\omega,x_3,x_1)\hat{C^{(1)}}(\omega,x_3,x_2)e^{-\mathrm{i}\omega\tau}d\omega = C^{(2)}(\tau,x_1,x_2) \end{aligned} \tag{3.5}$$

as $T' \to \infty$, and the convergence could be as fast as we need as long as $\hat{F}^\varepsilon$ is smooth enough. So it suffers to prove that

$$\langle C_{T,T'}\rangle \to \int_{-T'}^{T'} C^{(1)}(\tau',x_3,x_1)C^{(1)}(\tau'+\tau,x_3,x_2)d\tau'.$$

In detail, we have,

$$\begin{aligned}
&\langle C_{T,T'}\rangle - \int_{-T'}^{T'} C^{(1)}(\tau',x_3,x_1)C^{(1)}(\tau'+\tau,x_3,x_2)d\tau' \\
=&\int_{-T'}^{T'}\Big(\langle C_T(\tau',x_3,x_1)C_T(\tau'+\tau,x_3,x_2)\rangle - \langle C_T(\tau',x_3,x_1)\rangle\langle C_T(\tau'+\tau,x_3,x_2)\rangle\Big)d\tau' \\
=&\int_{-T'}^{T'}\mathrm{Cov}\Big(C_T(\tau',x_3,x_1)C_T(\tau'+\tau,x_3,x_2)\Big)d\tau',
\end{aligned} \tag{3.6}$$

where Cov stands for the covariance of two random variables. Meanwhile,

$$\begin{aligned}
&\mathrm{Cov}\Big(C_T(\tau',x_3,x_1)C_T(\tau'+\tau,x_3,x_2)\Big) \\
=&\frac{1}{T^2}\int_0^T\int_0^T dtdt'\int dsds'dudu'\int dy_1dy_1'dy_2dy_2' \\
&\times G(s,x_3,y_1)\,G(u+\tau',x_1,y_2)\,G(s',x_3,y_1')\,G(u'+\tau'+\tau,x_2,y_2') \\
&\times\Big[\langle n^\varepsilon(t-s,y_1)n^\varepsilon(t-u,y_2)n^\varepsilon(t'-s',y_1')n^\varepsilon(t'-u',y_2')\rangle \\
&-\langle n^\varepsilon(t-s,y_1)n^\varepsilon(t-u,y_2)\rangle\langle n^\varepsilon(t'-s',y_1')n^\varepsilon(t'-u',y_2')\rangle\Big].
\end{aligned} \tag{3.7}$$

Since n^ε is a zero mean Gaussian process in time, by Isserlis' Theorem [17], combining (2.2), we have

$$\begin{aligned}
&\Big[\langle n^\varepsilon(t-s,y_1)n^\varepsilon(t-u,y_2)n^\varepsilon(t'-s',y_1')n^\varepsilon(t'-u',y_2')\rangle \\
&-\langle n^\varepsilon(t-s,y_1)n^\varepsilon(t-u,y_2)\rangle\langle n^\varepsilon(t'-s',y_1')n^\varepsilon(t'-u',y_2')\rangle\Big] \\
=&\langle n^\varepsilon(t-s,y_1)n^\varepsilon(t'-s',y_1')\rangle\langle n^\varepsilon(t-u,y_2)n^\varepsilon(t'-u',y_2')\rangle \\
&+\langle n^\varepsilon(t-s,y_1)n^\varepsilon(t'-u',y_2')\rangle\langle n^\varepsilon(t-u,y_2)n^\varepsilon(t'-s',y_1')\rangle \\
=&F^\varepsilon(t-t'-s+s')F^\varepsilon(t-t'-u+u')\theta(y_1)\delta(y_1-y_1')\theta(y_2)\delta(y_2-y_2') \\
&+F^\varepsilon(t-t'-s+u')F^\varepsilon(t-t'-u+s')\theta(y_1)\delta(y_1-y_2')\theta(y_2)\delta(y_2-y_1').
\end{aligned} \tag{3.8}$$

Here note that,

$$\begin{aligned}
&\frac{1}{T^2}\int_0^T\int_0^T F^\varepsilon(t-t'-s)F^\varepsilon(t-t'-u)dtdt' \\
=&\frac{1}{4\pi^2T^2}\int_0^T\int_0^T dtdt'\int d\omega d\omega'\hat{F}^\varepsilon(\omega)\hat{F}^\varepsilon(\omega')e^{-\mathrm{i}\omega(t-t'-s)}e^{\mathrm{i}\omega'(t-t'-u)} \\
=&\frac{1}{4\pi^2T^2}\int d\omega d\omega'\hat{F}^\varepsilon(\omega)\hat{F}^\varepsilon(\omega')e^{\mathrm{i}\omega s-\mathrm{i}\omega' u}\int_0^T dte^{\mathrm{i}(\omega'-\omega)t}\int_0^T dt'e^{\mathrm{i}(\omega-\omega')t'}.
\end{aligned} \tag{3.9}$$

On the other hand, setting $s=(t/T-1/2)$, and noting the fact that the inverse Fourier transform of $\mathrm{sinc}(\omega/2)$ is $\mathrm{rect}(t)$, where

$$\mathrm{rect}(t)=\begin{cases}1, & |t|\le\frac{1}{2},\\ 0, & |t|>\frac{1}{2},\end{cases}$$

we have

$$\int_0^T dte^{\mathrm{i}(\omega'-\omega)t}=\int_{-\frac{1}{2}}^{\frac{1}{2}} dsTe^{\mathrm{i}T(\omega'-\omega)(s+\frac{1}{2})}=e^{\mathrm{i}(\omega'-\omega)\frac{T}{2}}\mathrm{sinc}\left(\frac{(\omega'-\omega)T}{2}\right). \tag{3.10}$$

Similarly, we have

$$\int_0^T dt'e^{\mathrm{i}(\omega-\omega)t'}=e^{\mathrm{i}(\omega-\omega')\frac{T}{2}}\mathrm{sinc}\left(\frac{(\omega-\omega')T}{2}\right). \tag{3.11}$$

Substituting (3.10)-(3.11) into (3.9), we have

$$\begin{aligned}&\frac{1}{T^2}\int_0^T\int_0^T F^\varepsilon(t-t'-s)F^\varepsilon(t-t'-u)dtdt'\\ =&\frac{1}{4\pi^2}\int d\omega d\omega'\hat{F}^\varepsilon(\omega)\hat{F}^\varepsilon(\omega')\mathrm{sinc}^2\left(\frac{(\omega-\omega')T}{2}\right)e^{\mathrm{i}\omega s-\mathrm{i}\omega' u}.\end{aligned} \tag{3.12}$$

Combining (3.7), (3.8) and (3.12), we obtain

$$\begin{aligned}&\mathrm{Cov}\Big(C_T(\tau',x_3,x_1)C_T(\tau'+\tau,x_3,x_2)\Big)\\ =&\frac{1}{4\pi^2}\int dy_1dy_2\theta(y_1)\theta(y_2)\int d\omega d\omega'\hat{F}^\varepsilon(\omega)\hat{F}^\varepsilon(\omega')e^{\mathrm{i}\omega'\tau}\\ &\times\hat{G}(\omega,x_3,y_1)\overline{\hat{G}}(\omega,x_3,y_1)\overline{\hat{G}}(\omega',x_2,y_2)\hat{G}(\omega',x_1,y_2)\mathrm{sinc}^2\left(\frac{(\omega-\omega')T}{2}\right)\\ &+\frac{1}{4\pi^2}\int dy_1dy_2\theta(y_1)\theta(y_2)\int d\omega d\omega'\hat{F}^\varepsilon(\omega)\hat{F}^\varepsilon(\omega')e^{\mathrm{i}(\omega+\omega')\tau'+i\omega'\tau}\\ &\times\hat{G}(\omega,x_3,y_1)\overline{\hat{G}}(\omega',x_1,y_2)\hat{G}(\omega',x_3,y_2)\overline{\hat{G}}(\omega,x_2,y_1)\mathrm{sinc}^2\left(\frac{(\omega-\omega')T}{2}\right).\end{aligned} \tag{3.13}$$

For the first part of (3.13) on the right-hand side, let $\omega_1=(\omega-\omega')T/2,\omega_2=(\omega+\omega')/2$, we have

$$
\begin{aligned}
&\frac{1}{4\pi^2}\int dy_1dy_2\theta(y_1)\theta(y_2)\int d\omega d\omega'\hat{F}^\varepsilon(\omega)\hat{F}^\varepsilon(\omega')e^{\mathrm{i}\omega'\tau}\\
&\times\hat{G}(\omega,x_3,y_1)\overline{\hat{G}}(\omega,x_3,y_1)\overline{\hat{G}}(\omega',x_1,y_2)\hat{G}(\omega',x_2,y_2)\,\mathrm{sinc}^2\left(\frac{(\omega-\omega')T}{2}\right)\\
=&\frac{1}{4\pi^2}\frac{2}{T}\int dy_1dy_2\theta(y_1)\theta(y_2)\int d\omega_1d\omega_2\\
&\times\hat{F}^\varepsilon\left(\omega_2+\frac{\omega_1}{T}\right)\hat{F}^\varepsilon\left(\omega_2-\frac{\omega_1}{T}\right)e^{\mathrm{i}(\omega_2-\omega_1/T)\tau}\\
&\times\hat{G}\left(\omega_2+\frac{\omega_1}{T},x_3,y_1\right)\overline{\hat{G}}\left(\omega_2+\frac{\omega_1}{T},x_3,y_1\right)\\
&\times\overline{\hat{G}}\left(\omega_2-\frac{\omega_1}{T},x_1,y_2\right)\hat{G}\left(\omega_2-\frac{\omega_1}{T},x_2,y_2\right)\mathrm{sinc}^2(\omega_1).
\end{aligned}\tag{3.14}
$$

As $T\to\infty$,

$$
\begin{aligned}
&(3.14)\to\frac{1}{2\pi^2T}\int dy_1dy_2\theta(y_1)\theta(y_2)\int d\omega_2\hat{F}^\varepsilon(\omega_2)\hat{F}^\varepsilon(\omega_2)e^{\mathrm{i}\omega_2\tau}\\
&\times\hat{G}(\omega_2,x_3,y_1)\overline{\hat{G}}(\omega_2,x_3,y_1)\overline{\hat{G}}(\omega_2,x_1,y_2)\hat{G}(\omega_2,x_2,y_2)\int d\omega_1\mathrm{sinc}^2(\omega_1)\\
=&\frac{1}{2\pi T}\int dy_1dy_2\theta(y_1)\theta(y_2)\int d\omega_2\hat{F}^\varepsilon(\omega_2)\hat{F}^\varepsilon(\omega_2)e^{\mathrm{i}\omega_2\tau}\\
&\times\hat{G}(\omega_2,x_3,y_1)\overline{\hat{G}}(\omega_2,x_3,y_1)\overline{\hat{G}}(\omega_2,x_1,y_2)\hat{G}(\omega_2,x_2,y_2).
\end{aligned}
$$

Similarly we could deal with another part of (3.13) on the right-hand side. Thus back to (3.6), we get that

$$
\langle C_{T,T'}\rangle-\int_{-T'}^{T'}C^{(1)}(\tau',x_3,x_1)C^{(1)}(\tau'+\tau,x_3,x_2)d\tau'\lesssim\frac{T'}{T}.\tag{3.15}
$$

Since $T'\lesssim T^\alpha$ and $0<\alpha<1$, we get,

$$
\langle C_{T,T'}\rangle\to\int_{-T'}^{T'}C^{(1)}(\tau',x_3,x_1)C^{(1)}(\tau'+\tau,x_3,x_2)d\tau',
$$

with convergent rate of order $\mathcal{O}(T^{-1+\alpha})$. Combining (3.5), we could conclude that,

$$
\langle C_{T,T'}(\tau,x_1,x_2)\rangle\to C^{(2)}(\tau,x_1,x_2),\tag{3.16}
$$

with convergent rate of order $\mathcal{O}(T^{-1+\alpha})$.

Next, we proof that $C_{T,T'}$ converge to its expectation $\langle C_{T,T'}\rangle$, which suffers to prove that the variance of $C_{T,T'}$ converge to zero. Indeed, we have

$$
\begin{aligned}
&\text{Cov}(C_{T,T'}(\tau,x_1,x_2)C_{T,T'}(\tau+\Delta\tau,x_1,x_2))\\
=&\frac{1}{T^4}\int_{-T'}^{T'}\int_{-T'}^{T'}d\tau_1 d\tau_2\int_0^T\int_0^T\int_0^T\int_0^T dt_1dt_2dt_3dt_4\\
&\times\int ds_1\cdots ds_8\int dy_1\cdots dy_8\\
&\times G(s_1,x_3,y_1)G(s_2+\tau_1,x_1,y_2)G(s_3,x_3,y_3)G(s_4+\tau_1+\tau,x_2,y_4)\\
&\times G(s_5,x_3,y_5)G(s_6+\tau_2,x_1,y_6)G(s_7,x_3,y_7)G(s_8+\tau_2+\tau+\Delta\tau,x_2,y_8)\\
&\times\big[\langle n^\varepsilon(t_1-s_1)n^\varepsilon(t_1-s_2)n^\varepsilon(t_2-s_3)n^\varepsilon(t_2-s_4)\\
&\times n^\varepsilon(t_3-s_5)n^\varepsilon(t_3-s_6)n^\varepsilon(t_4-s_7)n^\varepsilon(t_4-s_8)\rangle\\
&-\langle n^\varepsilon(t_1-s_1)n^\varepsilon(t_1-s_2)n^\varepsilon(t_2-s_3)n^\varepsilon(t_2-s_4)\rangle\\
&\times\langle n^\varepsilon(t_3-s_5)n^\varepsilon(t_3-s_6)n^\varepsilon(t_4-s_7)n^\varepsilon(t_4-s_8)\rangle\big].
\end{aligned}
$$

Again by using Isserlis' Theorem, the right hand of the equation above could be rewritten as a sum of 96 terms. For each of these 96 terms, through a similar process as in the first part of the proof, we could prove that it is of order $\mathcal{O}(T'^2/T^2)$. Let $\Delta\tau=0$, we obtain

$$\text{Var}(C_{T,T'}(\tau,x_1,x_2))\lesssim\frac{T'^2}{T^2},$$

where Var stands for the variance. Since $T'\lesssim T^\alpha$ and $0<\alpha<1$, we could conclude that, as $T,T'\to\infty$,

$$C_{T,T'}\to\langle C_{T,T'}\rangle, \tag{3.17}$$

with convergent rate of order $\mathcal{O}(T^{-1+\alpha})$. A combination of (3.16) and (3.17) finally prove our theorem. □

Theorem 2.1 and Theorem 3.1 show that both the first-order and the second-order cross correlation are statistically stable quantities. However, as shown in Theorem 3.1, the convergent rate for the second-order cross correlation is of order $\mathcal{O}(T^{-1+\alpha})$. When $0<\alpha<1/2$, this convergent rate is faster than that for the first-order cross correlation, which is only of order $\mathcal{O}(T^{-1/2})$. In some sense, this higher order convergent rate means that the second-order cross correlation might be more stable than the first-order cross correlation, when the observation time is given and fixed.

4 Homogeneous medium

As shown in Theorem 2.2 and Theorem 2.3, the first-order cross correlation contains both the travel time information and the target reflector information. Since the second-order cross correlation is defined based on the first-order one, it is natural to consider that it also has the similar properties. In this section, we will firstly analyze the behavior of the second-order cross correlation in the homogeneous medium, both with and without a target reflector.

To begin with, here and in what after, we introduce the symbol $a \to b \to c$, which means the condition that there is a wave ray starting from a, passing b, and finally reaching c.

4.1 Homogeneous medium without reflectors

When there is no reflectors, the coefficient $c(x)$ in (2.1) is a constant c_0. In this case, we know from Theorem 2.2 that the first-order cross correlation could reveal the travel time of wave signals between sensors. In fact, the second-order cross correlation also has such a property.

Theorem 4.1. *As ε tends to zero, if there exists wave rays satisfying condition $y_1 \to x_3 \to x_1$, $y_2 \to x_3 \to x_3$, where y_i are in the support of the random sources, the second-order cross correlation $C^{(2)}$ has a singular component at $\tau = \tau(x_2,x_3) - \tau(x_1,x_3)$.*

Proof. According to [18], the corresponding Green's function in this case could be written as,

$$\hat{G}(\omega,x,y) = \frac{e^{\mathrm{i}\frac{\omega}{c_0}|x-y|}}{4\pi|x-y|} = \frac{e^{\mathrm{i}\omega\tau(x,y)}}{4\pi|x-y|}, \tag{4.1}$$

where $\tau(x,y) = |x-y|/c_0$ stands for the travel time of waves between x,y. Substituting it into (3.4), noting the scaling (2.4), we obtain,

$$\begin{aligned} C^{(2)}(\tau,x_1,x_2) =& \frac{1}{2^9\pi^5}\int dy_1 dy_2 \theta(y_1)\theta(y_2)\int d\omega\,[F^\varepsilon(\omega)]^2 e^{\mathrm{i}\omega\mathcal{T}} \\ &\times \frac{1}{|x_3-y_1||x_1-y_1||x_3-y_2||x_2-y_2|} \\ =& \frac{\varepsilon}{2^9\pi^5}\int dy_1 dy_2 \theta(y_1)\theta(y_2)\int d\omega F^2(\omega) e^{\mathrm{i}\frac{\omega}{\varepsilon}\mathcal{T}} \\ &\times \frac{1}{|x_3-y_1||x_1-y_1||x_3-y_2||x_2-y_2|}, \end{aligned} \tag{4.2}$$

where the phase term is given by,

$$\omega\mathcal{T} = \omega\Big[\tau(x_3,y_1) - \tau(x_1,y_1) - \tau(x_3,y_2) + \tau(x_2,y_2) - \tau\Big]. \tag{4.3}$$

Note that the existence of small parameter ε results in that the integral on the right-hand side of (4.2) becomes an oscillatory integral with high frequency. Thus by stationary phase method [6], the dominant contribution of (4.2) comes from the stationary points of the phase which satisfy

$$\partial_\omega(\omega\mathcal{T}) = 0, \qquad \nabla_{y_1}(\omega\mathcal{T}) = 0, \qquad \nabla_{y_2}(\omega\mathcal{T}) = 0.$$

This implies that

$$\tau=\tau(x_3,y_1)-\tau(x_1,y_1)-\tau(x_3,y_2)+\tau(x_2,y_2), \tag{4.4}$$

$$\nabla_{y_1}\tau(x_3,y_1)=\nabla_{y_1}\tau(x_1,y_1), \qquad \nabla_{y_2}\tau(x_3,y_2)=\nabla_{y_2}\tau(x_2,y_2). \tag{4.5}$$

By Lemma B.1 in the appendix in [14], (4.5) requires the conditions $y_1\to x_3\to x_1$ and $y_2\to x_3\to x_2$. Combining (4.4), we get that $C^{(2)}$ has a singular component at $\tau=\tau(x_2,x_3)-\tau(x_1,x_3)$. □

For Theorem 4.1, if we specially choose the locations of the sensors x_1,x_2,x_3, such that these three points lie on the same wave ray issuing from the source region, then the second-order cross correlation has a singular component at $\tau=\pm\tau(x_1,x_2)$. More precisely, if the points are aligned along the ray as $y\to x_3\to x_1\to x_2$, the singular component is at $\tau=\tau(x_1,x_2)$. While if the points are aligned along the ray as $y\to x_3\to x_2\to x_1$, the singular component is at $\tau=-\tau(x_1,x_2)$. This result agrees with that of the first-order cross correlation, which reveals the travel time between sensors x_1,x_2. The following example considers a homogeneous case without reflectors, which gives a verification to Theorem 4.1.

Example 4.1. For simplicity, here and in what after, all the examples are taken at the plane $z=0$ in $\mathbb{R}^3$. Here we consider the domain $[-0.5,0.5]^2$. Specially we set the random noise sources at the domain $[-0.5,-0.4]\times[0.5,0.5]$, set the sensors at $x_1=(0,0),x_2=(0.2,0)$, and set the auxiliary sensor at $x_3=(-0.2,0)$. By simulating the forward problem (2.1) with the coefficient $c(x)=c_0=0.05$, we collect the wave signals at all these sensors. Fig. 2 illustrates the wave signals collected by the three sensors for a total observation time $T=100$s. As

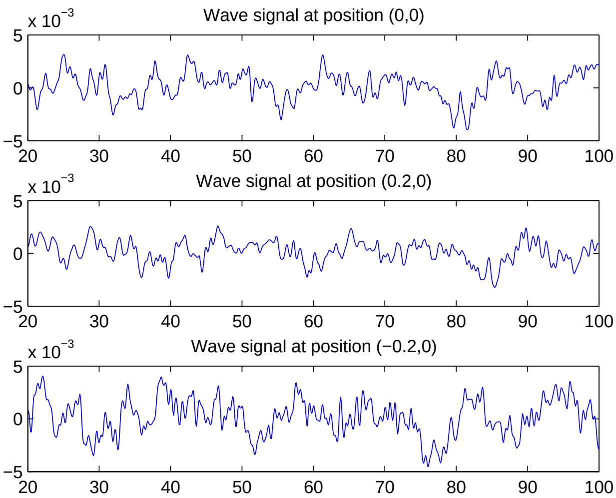

Figure 2: Wave signals at $(0,0),(0.2,0),(-0.2,0)$ for a total observation time $T=100$s.

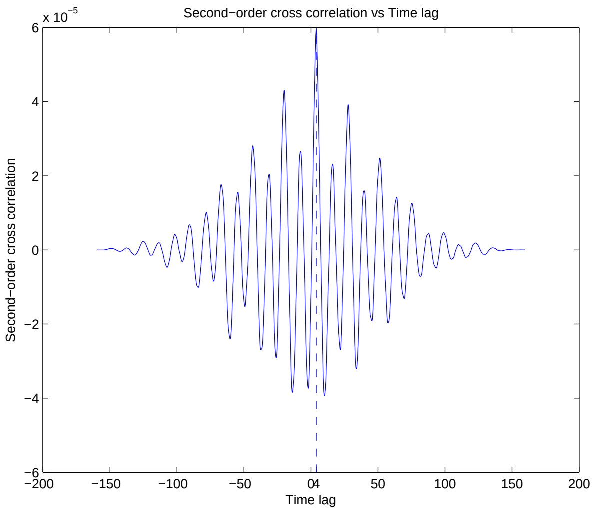

Figure 3: The second-order cross correlation between $x_1=(0,0)$ and $x_2=(0.2,0)$.

expected, we could not get any useful information from these signals. Fig. 3 illustrates the second-order cross correlation between x_1 and x_2. We could find that it has a singular component at $\tau=4s$. On the other hand, according to Theorem 4.1, under the setting of this example, this second-order cross correlation should have a singular component at $\tau=\tau(x_1,x_2)=4$s. The agreement of two aspects verifies Theorem 4.1.

4.2 Homogeneous medium with a point reflector

Now we consider the case with a point reflector. As stated in Section 2, the space configuration could be either a daylight configuration or a backlight configuration, based on the relative locations of sources, sensors and reflector. In this part, for simplicity, we only consider the backlight case. Indeed, we consider the space configuration as shown in Fig. 4, which means that the reflector is located between the sources and the sensors. Here and in what follows, in the sensor area, we further assume that the auxiliary sensor x_3 is on the left-hand side of the sensors x_1,x_2, or in another word, the auxiliary sensor x_3 is closer to the random sources.

Since we consider the reflector here, the coefficient $c(x)$ in (2.1) is no longer a constant. Under the fact that the reflector is a point reflector, it could be assumed that the coefficient satisfies,

$$\frac{1}{c^2(x)}=\frac{1}{c_0^2}[1+V_r(x)], \tag{4.6}$$

where c_0 is the background velocity. $V_r(x)$ characterizes the influence of the reflector on the total wave speed, which satisfies,

$$V_r(x)=\sigma_r l_r^3\delta(x-z_r), \tag{4.7}$$

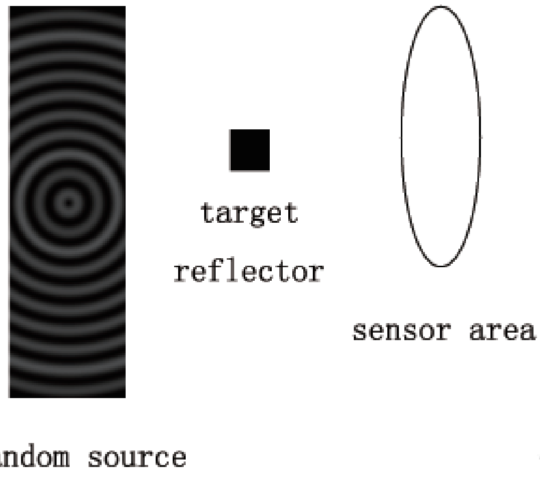

Figure 4: Space configuration. From left to right are source region, reflector, and sensor area, respectively.

where $0<\sigma_r\ll 1$ is the reflectivity, l_r^3 characterizes the volume of the reflector, and z_r is the location of the reflector. By Born approximation [9], the corresponding Green's function in this case could be written as,

$$\hat{G}(\omega,x,y)=\hat{G}_0(\omega,x,y)+\hat{G}_r(\omega,x,y)+\mathcal{O}(\sigma_r^2), \tag{4.8}$$

where $\hat{G}_0$ is the Green's function without reflector, which is given by (4.1). $\hat{G}_r$ is the first-order approximation term due to the point reflector, which is given by,

$$\hat{G}_r(\omega,x,y)=\frac{\omega^2}{c_0^2}\sigma_r l_r^3\hat{G}_0(\omega,x,z_r)\hat{G}_0(\omega,z_r,y). \tag{4.9}$$

The following theorem shows the behavior of the second-order cross correlation in this case.

Theorem 4.2. *Given the space configuration as Fig. 4, assume that there is a point reflector at z_r, with reflectivity $0<\sigma_r\ll 1$. As ε tends to zero, besides the singular component of order $\mathcal{O}(1)$ (with respect to σ_r) shown in Theorem 4.1, the second-order cross correlation $C^{(2)}$ also has a singular component at $\tau=\tau(x_2,z_r)-\tau(x_1,z_r)$, of order $\mathcal{O}(\sigma_r)$, with sensor locations specially chosen.*

Proof. Substituting the Green's function (4.8) into (3.4), noting the scaling (2.4), we obtain,

$$\begin{aligned}C^{(2)}(\tau,x_1,x_2)=&\frac{\varepsilon}{2\pi}\int dy_1dy_2\theta(y_1)\theta(y_2)\int d\omega F^2(\omega)e^{-\mathrm{i}\frac{\omega}{\varepsilon}\tau}\\&\times\left[\hat{G}_0\left(\frac{\omega}{\varepsilon},x_3,y_1\right)+\hat{G}_r\left(\frac{\omega}{\varepsilon},x_3,y_1\right)\right]\left[\overline{\hat{G}}_0\left(\frac{\omega}{\varepsilon},x_1,y_1\right)+\overline{\hat{G}}_r\left(\frac{\omega}{\varepsilon},x_1,y_1\right)\right]\\&\times\left[\overline{\hat{G}}_0\left(\frac{\omega}{\varepsilon},x_3,y_2\right)+\overline{\hat{G}}_r\left(\frac{\omega}{\varepsilon},x_3,y_2\right)\right]\left[\hat{G}_0\left(\frac{\omega}{\varepsilon},x_2,y_2\right)+\hat{G}_r\left(\frac{\omega}{\varepsilon},x_2,y_2\right)\right]\\&+\mathcal{O}(\sigma_r^2).\end{aligned} \tag{4.10}$$

For the equation above, the term of order $\mathcal{O}(1)$ is the same as that in the proof of Theorem 4.1, so here we just consider the terms of order $\mathcal{O}(\sigma_r)$. For one term among them, denoted as $C^{(2)}_{\sigma_r(1)}$, we have,

$$\begin{aligned}
C^{(2)}_{\sigma_r(1)}(\tau,x_1,x_2)=&\frac{\varepsilon}{2\pi}\int dy_1dy_2\theta(y_1)\theta(y_2)\int d\omega F^2(\omega)e^{-\mathrm{i}\frac{\omega}{\varepsilon}\tau}\\
&\times\hat{G}_r\Big(\frac{\omega}{\varepsilon},x_3,y_1\Big)\overline{\hat{G}_0}\Big(\frac{\omega}{\varepsilon},x_1,y_1\Big)\overline{\hat{G}_0}\Big(\frac{\omega}{\varepsilon},x_3,y_2\Big)\hat{G}_0\Big(\frac{\omega}{\varepsilon},x_2,y_2\Big)\\
=&\frac{\sigma_r l_r^3}{2\pi\varepsilon c_0^2}\int dy_1dy_2\theta(y_1)\theta(y_2)\int d\omega\omega^2F^2(\omega)e^{-\mathrm{i}\frac{\omega}{\varepsilon}\tau}\\
&\times\hat{G}_0\Big(\frac{\omega}{\varepsilon},x_3,z_r\Big)\hat{G}_0\Big(\frac{\omega}{\varepsilon},z_r,y_1\Big)\overline{\hat{G}_0}\Big(\frac{\omega}{\varepsilon},x_1,y_1\Big)\overline{\hat{G}_0}\Big(\frac{\omega}{\varepsilon},x_3,y_2\Big)\hat{G}_0\Big(\frac{\omega}{\varepsilon},x_2,y_2\Big)\\
=&\frac{\sigma_r l_r^3}{2^{11}\pi^6\varepsilon c_0^2}\int dy_1dy_2\theta(y_1)\theta(y_2)\int d\omega\omega^2F^2(\omega)e^{\mathrm{i}\frac{\omega}{\varepsilon}\mathcal{T}}\\
&\times\frac{1}{|x_3-z_r||z_r-y_1||x_1-y_1||x_3-y_2||x_2-y_2|},
\end{aligned}\tag{4.11}$$

where the phase term is given by,

$$\omega\mathcal{T}=\omega\Big[\tau(x_3,z_r)+\tau(z_r,y_1)-\tau(x_1,y_1)-\tau(x_3,y_2)+\tau(x_2,y_2)-\tau\Big].\tag{4.12}$$

Again by the stationary phase method, the dominant contribution of (4.11) comes from the stationary points of the phase which satisfy

$$\tau=\tau(x_3,z_r)+\tau(z_r,y_1)-\tau(x_1,y_1)-\tau(x_3,y_2)+\tau(x_2,y_2),\tag{4.13}$$

$$\nabla_{y_1}\tau(z_r,y_1)=\nabla_{y_1}\tau(x_1,y_1),\qquad \nabla_{y_2}\tau(x_3,y_2)=\nabla_{y_2}\tau(x_2,y_2).\tag{4.14}$$

(4.14) implies the conditions $y_1\to z_r\to x_1$ and $y_2\to x_3\to x_2$. Combining (4.13), we could conclude that the integral (4.11) has a singular component at $\tau=\tau(x_3,z_r)+\tau(x_2,z_r)-\tau(x_1,z_r)$. Specially if we choose the location of sensor x_3 such that z_r,x_2,x_3 could be on the same ray issuing from the source region, then the singular component will appear at $\tau=\tau(x_2,z_r)-\tau(x_1,z_r)$. For the other terms of order $\mathcal{O}(\sigma_r)$ in (4.10), we could make the similar analysis. □

Theorem 4.2 tells us that, under the backlight space configuration with a point reflector, the second-order cross correlation $C^{(2)}$ has a singular component of order $\mathcal{O}(\sigma_r)$ at $\tau=\tau(x_2,z_r)-\tau(x_1,z_r)$ if we specially choose the location of auxiliary sensor x_3. Compared to Theorem 2.3, we see that this result agrees with that of the first-order cross correlation. By Theorem 4.2, we know the fact that the second-order cross correlation contains the location information of the point reflector, which enable us to realize the passive imaging using the second-order cross correlation in the homogeneous medium. The following example consider a homogeneous case with a point reflector, which gives a verification to Theorem 4.2.

Example 4.2. Here we take the similar setting as that in Example 4.1. The only difference is that we additionally set a point reflector at $z_r=(-0.25,0)$. Specially, in this example we set the sensors at $x_1=(0.2,-0.1), x_2=(0.2,0)$. And we set the auxiliary sensor at $x_3=(-0.1,0)$. Fig. 5 illustrates the wave signals collected by the three sensors for a total observation time $T=100$s. We still could not get any useful information on the location of the target reflector from these signals. Fig. 6 illustrates the second-order cross correlation of order $\mathcal{O}(\sigma_r)$ between x_1 and x_2. We could find that it has a singular component at about $\tau=-0.2$s. On the other hand, according to Theorem 4.2, under the setting of this example, this second-order cross correlation of order $\mathcal{O}(\sigma_r)$ should have a singular component at $\tau=\tau(x_2,z_r)-\tau(x_1,z_r)\approx-0.22$s. The agreement of two aspects verifies Theorem 4.2.

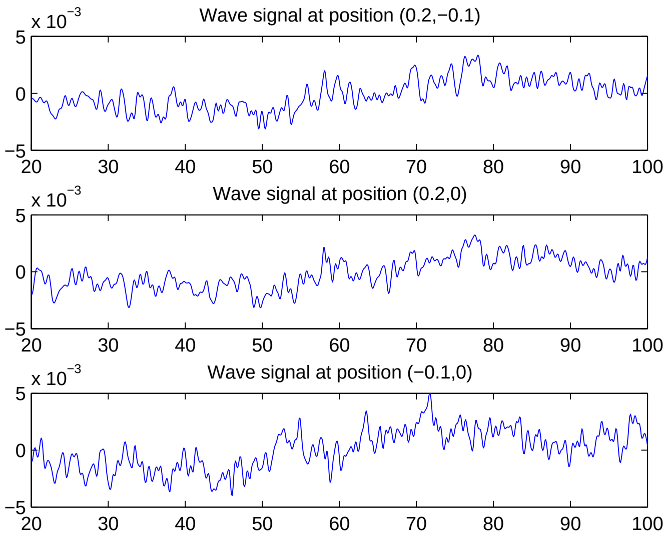

Figure 5: Wave signals at $(0.2,-0.1),(0.2,0),(-0.1,0)$ for a total observation time $T=100$s.

5 Scattering medium

In this section, we will discuss the case with scattering medium. For the simplicity of analysis, we assume that the scattering medium is located behind the sensor area, i.e., we consider the space configuration as shown in Fig. 7

Firstly, we give a model to the stochastic scattering medium. We assume that the wave speed consists of a homogeneous background velocity c_0 and small stochastic fluctuations. Mathematically, we could assume that the coefficient $c(x)$ in (2.1) satisfies,

$$\frac{1}{c^2(x)}=\frac{1}{c_0^2}(1+V(x)), \tag{5.1}$$

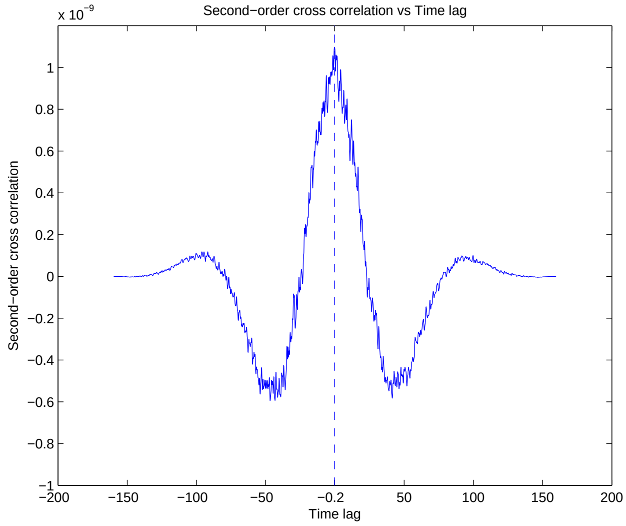

Figure 6: The second-order cross correlation of order $\mathcal{O}(\sigma_r)$ between $x_1=(0.2,-0.1)$ and $x_2=(0.2,0)$.

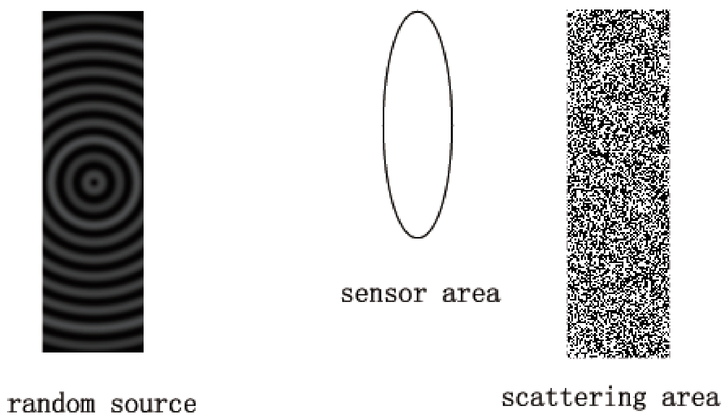

Figure 7: Space configuration. From left to right are source region, sensor area, and scattering area, respectively.

where $V(x)$ is a zero-mean Gaussian process, which has the statistic property,

$$\mathbb{E}[V(x)V(x')]=\sigma_s^2 R(x,x'). \tag{5.2}$$

Here $0<\sigma_s\ll 1$ stands for the standard variance of the stochastic fluctuations. Further more, we assume that the stochastic fluctuations are Gaussian white noise, so that we have,

$$\mathbb{E}[V(x)V(x')]=\sigma_s^2 l_s^3 \rho(x)\delta(x-x'), \tag{5.3}$$

where l_s is the characteristic correlation length, and $\rho(x)$ characterizes the support of the scattering area.

5.1 Second-order coda cross correlation

To give a better analysis of the influence of the scattering medium on the cross correlation, in this part, we introduce the coda cross correlation. Given any two points x_1,x_2, we define the first-order coda cross correlation as follows,

$$C_{T,coda}(\tau,x_1,x_2)=C_T(\tau,x_1,x_2)-C_{0,T}(\tau,x_1,x_2), \tag{5.4}$$

where C_T is the first-order cross correlation in the scattering medium and $C_{0,T}$ is the first-order cross correlation in the corresponding homogeneous medium. Then, with the auxiliary sensor x_3, the second-order coda cross correlation could be defined by

$$C_{T,T',coda}(\tau,x_1,x_2)=\int_{-T'}^{T'} C_{T,coda}(\tau',x_3,x_1)C_{T,coda}(\tau'+\tau,x_3,x_2)d\tau'. \tag{5.5}$$

Note that, similarly as $C_{T,T'}$, $C_{T,T',coda}$ defined above also relies on the location of x_3. However, as stated in Remark 3.1, as long as there are enough auxiliary sensors, this influence could be eliminated by integration.

Meanwhile, as a direct generation of Theorem 3.1, for the second-order coda cross correlation, we have the following statistic property.

Theorem 5.1. *Given positive constant $\alpha<1$, as $T,T'\to\infty$ and $T'\lesssim T^\alpha$, the second-order coda cross correlation satisfies,*

$$C_{T,T',coda}(\tau,x_1,x_2)\to C^{(2)}_{coda}(\tau,x_1,x_2). \tag{5.6}$$

Here

$$\begin{aligned} C^{(2)}_{coda}(\tau,x_1,x_2)=&\frac{\varepsilon}{2\pi}\int dy_1dy_2\theta(y_1)\theta(y_2)\int d\omega F^2(\omega)e^{-\mathrm{i}\frac{\omega}{\varepsilon}\tau}\\ &\times\Big[\hat{G}(\frac{\omega}{\varepsilon},x_3,y_1)\overline{\hat{G}}(\frac{\omega}{\varepsilon},x_1,y_1)-\hat{G}_0(\frac{\omega}{\varepsilon},x_3,y_1)\overline{\hat{G}}_0(\frac{\omega}{\varepsilon},x_1,y_1)\Big]\\ &\times\Big[\overline{\hat{G}}(\frac{\omega}{\varepsilon},x_3,y_2)\hat{G}(\frac{\omega}{\varepsilon},x_2,y_2)-\overline{\hat{G}}_0(\frac{\omega}{\varepsilon},x_3,y_2)\hat{G}_0(\frac{\omega}{\varepsilon},x_2,y_2)\Big], \end{aligned} \tag{5.7}$$

where $\hat{G}$ is the Green's function with the scattering medium, and $\hat{G}_0$ is the Green's function with the corresponding homogeneous medium. More precisely, the convergent rate is of order $\mathcal{O}(T^{-1+\alpha})$.

5.2 Scattering medium without reflectors

Similarly as the homogeneous case, we start from the Green's function. With the wave speed assumption (5.1)-(5.3), by Born approximation, the Green's function in this case could be written as,

$$\hat{G}(\omega,x,y)=\hat{G}_0(\omega,x,y)+\hat{G}_1(\omega,x,y)+\mathcal{O}(\sigma_s^2), \tag{5.8}$$

where $\hat{G}_0$ is the Green's function in the corresponding homogeneous case, which is given by (4.1), and $\hat{G}_1$ is the first-order approximation term due to the stochastic fluctuations, which is given by,

$$\hat{G}_1(\omega,x,y)=\frac{\omega^2}{c_0^2}\int \hat{G}_0(\omega,x,z)V(z)\hat{G}_0(\omega,z,y)dz. \tag{5.9}$$

The following theorem shows the behavior of the second-order coda cross correlation in this case.

Theorem 5.2. *Given the space configuration as Fig. 7, as ε tends to zero, the expectation of the second-order coda cross correlation $C^{(2)}_{coda}$ has a singular component at $\tau=\pm\tau(x_1,x_2)$ if and only if there is a ray, issuing from the source region and reflected by the scattering area, satisfies $y\to x_3\to z$, and $z\to x_1,x_2$. Here y is a certain point in the source region and z is a certain point in the scattering area. More precisely, if the reflected ray satisfies $z\to x_1\to x_2$, then the singular component appears at $\tau=\tau(x_1,x_2)$. While if the reflected ray satisfies $z\to x_2\to x_1$, the singular component appears at $\tau=-\tau(x_1,x_2)$.*

Proof. Substituting (5.8) into (5.7), noting the scaling (2.4), we obtain,

$$\begin{aligned} C^{(2)}_{coda}(\tau,x_1,x_2)=&\frac{\varepsilon}{2\pi}\int dy_1dy_2\theta(y_1)\theta(y_2)\int d\omega F^2(\omega)e^{-\mathrm{i}\frac{\omega}{\varepsilon}\tau}\\ &\times\Big[\hat{G}_0(\frac{\omega}{\varepsilon},x_3,y_1)\overline{\hat{G}}_1(\frac{\omega}{\varepsilon},x_1,y_1)+\hat{G}_1(\frac{\omega}{\varepsilon},x_3,y_1)\overline{\hat{G}}_0(\frac{\omega}{\varepsilon},x_1,y_1)\Big]\\ &\times\Big[\overline{\hat{G}}_0(\frac{\omega}{\varepsilon},x_3,y_2)\hat{G}_1(\frac{\omega}{\varepsilon},x_2,y_2)+\overline{\hat{G}}_1(\frac{\omega}{\varepsilon},x_3,y_2)\hat{G}_0(\frac{\omega}{\varepsilon},x_2,y_2)\Big]\\ &+\mathcal{O}(\sigma_s^3). \end{aligned} \tag{5.10}$$

Since $\hat{G}_1$ is the approximation term of order $\mathcal{O}(\sigma_s)$, the right-hand side of the equation above contains four terms of order $\mathcal{O}(\sigma_s^2)$. Without loss of generality, for one term among them, denoted as $C^{(2)}_{coda(1)}$, we have,

$$\begin{aligned}C^{(2)}_{coda(1)}(\tau,x_1,x_2)=&\frac{\varepsilon}{2\pi}\int dy_1dy_2\theta(y_1)\theta(y_2)\int d\omega F^2(\omega)e^{-\mathrm{i}\frac{\omega}{\varepsilon}\tau}\\&\times\hat{G}_0\left(\frac{\omega}{\varepsilon},x_3,y_1\right)\overline{\hat{G}}_1\left(\frac{\omega}{\varepsilon},x_1,y_1\right)\overline{\hat{G}}_0\left(\frac{\omega}{\varepsilon},x_3,y_2\right)\hat{G}_1\left(\frac{\omega}{\varepsilon},x_2,y_2\right)\\=&\frac{1}{2\pi\varepsilon^3c_0^4}\int dy_1dy_2\theta(y_1)\theta(y_2)\int d\omega\omega^4F^2(\omega)e^{-\mathrm{i}\frac{\omega}{\varepsilon}\tau}\int dzdz'\\&\times\hat{G}_0\left(\frac{\omega}{\varepsilon},x_3,y_1\right)\overline{\hat{G}}_0\left(\frac{\omega}{\varepsilon},x_1,z\right)V(z)\overline{\hat{G}}_0\left(\frac{\omega}{\varepsilon},z,y_1\right)\overline{\hat{G}}_0\left(\frac{\omega}{\varepsilon},x_3,y_2\right)\\&\times\hat{G}_0\left(\frac{\omega}{\varepsilon},x_2,z'\right)V(z')\hat{G}_0\left(\frac{\omega}{\varepsilon},z',y_2\right).\end{aligned}$$

By taking expectation on both sides of the equation above, and combining the statistic property (5.3), it yields,

$$\begin{aligned}\mathbb{E}\left[C^{(2)}_{coda(1)}(\tau,x_1,x_2)\right]=&\frac{\sigma_s^2l_s^3}{2\pi\varepsilon^3c_0^4}\int dy_1dy_2dz\theta(y_1)\theta(y_2)\rho(z)\int d\omega\omega^4F^2(\omega)e^{-\mathrm{i}\frac{\omega}{\varepsilon}\tau}\\&\times\hat{G}_0\left(\frac{\omega}{\varepsilon},x_3,y_1\right)\overline{\hat{G}}_0\left(\frac{\omega}{\varepsilon},x_1,z\right)\overline{\hat{G}}_0\left(\frac{\omega}{\varepsilon},z,y_1\right)\\&\times\overline{\hat{G}}_0\left(\frac{\omega}{\varepsilon},x_3,y_2\right)\hat{G}_0\left(\frac{\omega}{\varepsilon},x_2,z\right)\hat{G}_0\left(\frac{\omega}{\varepsilon},z,y_2\right)\\=&\frac{\sigma_s^2l_s^3}{2^{13}\pi^7\varepsilon^3c_0^4}\int dy_1dy_2dz\theta(y_1)\theta(y_2)\rho(z)\int d\omega\omega^4F^2(\omega)e^{\mathrm{i}\frac{\omega}{\varepsilon}\mathcal{T}}\\&\times\frac{1}{|x_3-y_1||x_1-z||z-y_1||x_3-y_2||x_2-z||z-y_2|},\end{aligned}\tag{5.11}$$

where the phase term is given by,

$$\omega\mathcal{T}=\omega\Big[\tau(x_3,y_1)-\tau(x_1,z)-\tau(z,y_1)-\tau(x_3,y_2)+\tau(x_2,z)+\tau(z,y_2)-\tau\Big].\tag{5.12}$$

Again by the stationary phase method, the dominant contribution of (5.11) comes from the stationary points of the phase which satisfy

$$\partial_\omega(\omega\mathcal{T})=0,\quad\nabla_{y_1}(\omega\mathcal{T})=0,\quad\nabla_{y_2}(\omega\mathcal{T})=0,\quad\nabla_z(\omega\mathcal{T})=0.$$

This implies that

$$\tau=\tau(x_3,y_1)-\tau(x_1,z)-\tau(z,y_1)-\tau(x_3,y_2)+\tau(x_2,z)+\tau(z,y_2),\tag{5.13}$$

$$\nabla_{y_1}\tau(x_3,y_1)=\nabla_{y_1}\tau(z,y_1),\quad\nabla_{y_2}\tau(x_3,y_2)=\nabla_{y_2}\tau(z,y_2),\tag{5.14}$$

$$\nabla_z\tau(x_1,z)+\nabla_z\tau(z,y_1)=\nabla_z\tau(x_2,z)+\nabla_z\tau(z,y_2).\tag{5.15}$$

(5.14)implies the condition that $y_1,y_2\to x_3\to z$. And (5.15) implies the condition that $z\to x_1,x_2$. Combining (5.13), we could conclude that (5.11) has a singular component at $\tau=\pm\tau(x_1,x_2)$. For the other terms of order $\mathcal{O}(\sigma_s^2)$ in (5.10), we could make the similar analysis. □

By Theorem 5.2, we know that when the medium is a scattering medium, we still could get the travel time information through the second-order coda cross correlation. This result is just the same for the second-order cross correlation in the homogeneous case. However, one thing different is that, due to the influence of the scattering medium, we no longer need that x_1,x_2,x_3 lie on the same ray issuing from the source region. This enable us to deal with the case when the axis formed by the two sensors x_1,x_2 is perpendicular to the main direction of the energy flux from the noise sources. The following example consider a scattering case without reflectors, which gives a verification to Theorem 5.2.

Example 5.1. For this example, we take the similar setting as that in Example 4.1. Besides, we set the scattering medium in the domain $[0.4,0.5]\times[-0.5,0.5]$. Specially, we set the sensors at $x_1=(0,0.1), x_2=(0.2,0.1)$. And we set the auxiliary sensor at $x_3=(-0.2,0)$. Fig. 8 illustrates the second-order coda cross correlation between x_1 and x_2. We could find that it has a singular component at $\tau=-4$s. On the other hand, according to Theorem 5.2, under the setting of this example, this second-order coda cross correlation should have a singular component at $\tau=-\tau(x_1,x_2)=-4$s. The agreement of two aspects verifies Theorem 5.3.

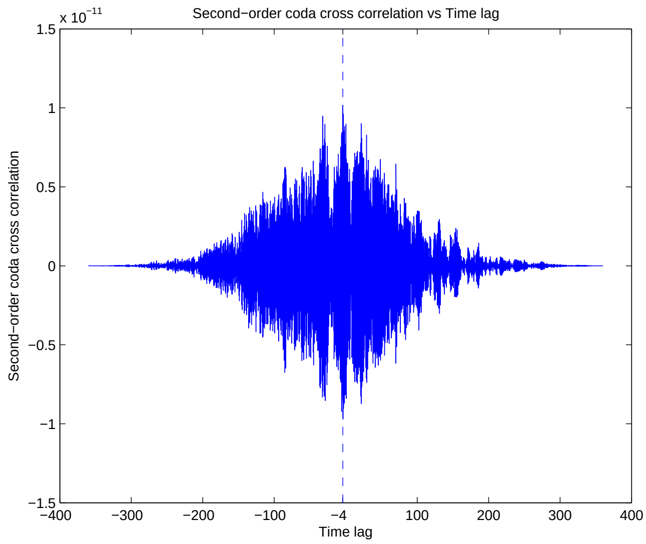

Figure 8: The second-order coda cross correlation between $x_1=(0,0.1)$ and $x_2=(0.2,0.1)$.

5.3 Scattering medium with a point reflector

Finally, we consider the scatting medium case with a point reflector. Similarly as the analysis in Section 4.2, we only consider a backlight configuration for simplicity. Indeed, we assume that the space configuration is set as Fig. 9.

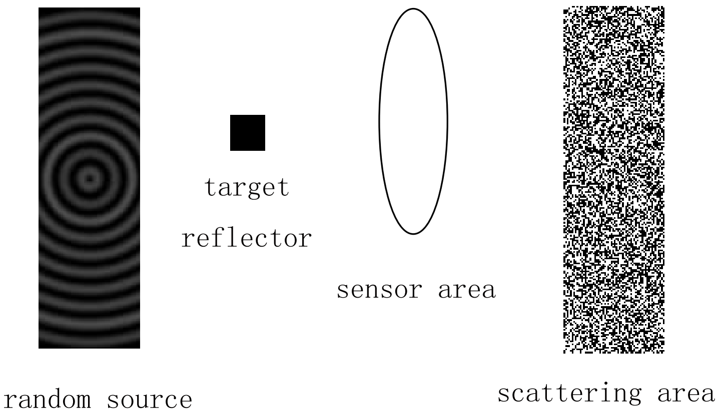

Figure 9: Space configuration. From left to right are source region, reflector, sensor area, and scattering area, respectively.

Similarly as the analysis in the previous sections, to analyze the behavior of the second-order coda cross correlation, we still begin with the Green's function. Combining (4.6), (5.1), we could assume that the coefficient $c(x)$ in (2.1) satisfies,

$$\frac{1}{c^2(x)}=\frac{1}{c_0^2}[1+V(x)+V_r(x)]. \tag{5.16}$$

Here the constant c_0 is the wave speed under the corresponding homogeneous case. $V(x)$ satisfies (5.3), which characterizes the influence of the scattering medium on the wave speed. $V_r(x)$ is given by (4.7), which characterizes the influence of the point reflector on the wave speed. Again with the help of Born approximation, we could write the Green's function as

$$\hat{G}(\omega,x,y)=\hat{G}_{clu}(\omega,x,y)+\frac{\omega^2\sigma_r l_r^3}{c_0^2}\hat{G}_{clu}(\omega,x,z_r)\hat{G}_{clu}(\omega,z_r,y)+\mathcal{O}(\sigma_r^2), \tag{5.17}$$

where $\hat{G}_{clu}$ is the Green's function under the scattering medium case without reflectors, which is given by (5.8).

Theorem 5.3. *Given the space configuration as Fig. 9, assume that there is a point reflector at z_r, with reflectivity $0<\sigma_r\ll 1$. As ε tends to zero, besides the singular component of order $\mathcal{O}(\sigma_s^2)$ shown in Theorem 5.2, the second-order coda cross correlation $C^{(2)}_{coda}$ also has singular components at $\tau=\pm(\tau(x_1,z_r)+\tau(x_2,z_r))$, of order $\mathcal{O}(\sigma_r\sigma_s^2)$, with sensor locations specially chosen.*

Proof. According to Theorem 5.1, with the scaling (2.4), the second-order coda cross correlation in this case could be expressed by,

$$\begin{aligned}C^{(2)}_{coda}(\tau,x_1,x_2)=&\frac{\varepsilon}{2\pi}\int dy_1dy_2\theta(y_1)\theta(y_2)\int d\omega F^2(\omega)e^{-\mathrm{i}\frac{\omega}{\varepsilon}\tau}\\&\times\Big[\hat{G}\Big(\frac{\omega}{\varepsilon},x_3,y_1\Big)\overline{\hat{G}}\Big(\frac{\omega}{\varepsilon},x_1,y_1\Big)-\hat{G}_{0,r}\Big(\frac{\omega}{\varepsilon},x_3,y_1\Big)\overline{\hat{G}}_{0,r}\Big(\frac{\omega}{\varepsilon},x_1,y_1\Big)\Big]\\&\times\Big[\overline{\hat{G}}\Big(\frac{\omega}{\varepsilon},x_3,y_2\Big)\hat{G}\Big(\frac{\omega}{\varepsilon},x_2,y_2\Big)-\overline{\hat{G}}_{0,r}\Big(\frac{\omega}{\varepsilon},x_3,y_2\Big)\hat{G}_{0,r}\Big(\frac{\omega}{\varepsilon},x_2,y_2\Big)\Big],\end{aligned}\tag{5.18}$$

where $\hat{G}$ is the Green's function given by (5.17), $\hat{G}_{0,r}$ is the Green's function with the corresponding homogeneous medium, which is given by (4.8). By a careful calculation, we obtain that (5.18) consists of two parts. Of one part, the Green's function part is,

$$\begin{aligned}&\Big[\hat{G}_0\Big(\frac{\omega}{\varepsilon},x_3,y_1\Big)\overline{\hat{G}}_1\Big(\frac{\omega}{\varepsilon},x_1,y_1\Big)+\hat{G}_1\Big(\frac{\omega}{\varepsilon},x_3,y_1\Big)\overline{\hat{G}}_0\Big(\frac{\omega}{\varepsilon},x_1,y_1\Big)\Big]\\&\times\Big[\overline{\hat{G}}_0\Big(\frac{\omega}{\varepsilon},x_3,y_2\Big)\hat{G}_1\Big(\frac{\omega}{\varepsilon},x_2,y_2\Big)+\overline{\hat{G}}_1\Big(\frac{\omega}{\varepsilon},x_3,y_2\Big)\hat{G}_0\Big(\frac{\omega}{\varepsilon},x_2,y_2\Big)\Big],\end{aligned}$$

which has been analyzed in Theorem 5.2. It has a singular component of order $\mathcal{O}(\sigma_s^2)$ at $\tau=\pm\tau(x_1,x_2)$. Another part has leading order $\mathcal{O}(\sigma_r\sigma_s^2)$. Without loss of generality, for one term in this part, denoted as $C^{(2)}_{codaII(1)}$, we have,

$$\begin{aligned}C^{(2)}_{codaII(1)}(\tau,x_1,x_2)=&\frac{\sigma_r l_r^3}{2\pi\varepsilon c_0^2}\int dy_1dy_2\theta(y_1)\theta(y_2)\int d\omega\omega^2F^2(\omega)e^{-\mathrm{i}\frac{\omega}{\varepsilon}\tau}\\&\times\hat{G}_0\Big(\frac{\omega}{\varepsilon},x_3,y_1\Big)\overline{\hat{G}}_1\Big(\frac{\omega}{\varepsilon},x_1,y_1\Big)\overline{\hat{G}}_0\Big(\frac{\omega}{\varepsilon},x_3,y_2\Big)\hat{G}_0\Big(\frac{\omega}{\varepsilon},x_2,z_r\Big)\hat{G}_1\Big(\frac{\omega}{\varepsilon},z_r,y_2\Big)\\=&\frac{\sigma_r l_r^3}{2\pi\varepsilon^5 c_0^6}\int dy_1dy_2\theta(y_1)\theta(y_2)\int d\omega\omega^6F^2(\omega)e^{-\mathrm{i}\frac{\omega}{\varepsilon}\tau}\int dzdz'\\&\times\hat{G}_0\Big(\frac{\omega}{\varepsilon},x_3,y_1\Big)\overline{\hat{G}}_0\Big(\frac{\omega}{\varepsilon},x_1,z\Big)V(z)\overline{\hat{G}}_0\Big(\frac{\omega}{\varepsilon},z,y_1\Big)\overline{\hat{G}}_0\Big(\frac{\omega}{\varepsilon},x_3,y_2\Big)\\&\times\hat{G}_0\Big(\frac{\omega}{\varepsilon},x_2,z_r\Big)\hat{G}_0\Big(\frac{\omega}{\varepsilon},z_r,z'\Big)V(z')\hat{G}_0\Big(\frac{\omega}{\varepsilon},z',y_2\Big).\end{aligned}$$

By taking expectation on both sides of the equation above, and combining the statistic property (5.3), it yields,

$$\begin{aligned}\mathbb{E}\Big[C^{(2)}_{codaII(1)}(\tau,x_1,x_2)\Big]=&\frac{\sigma_r\sigma_s^2 l_s^3 l_r^3}{2\pi\varepsilon^5 c_0^6}\int dy_1dy_2d\omega dz\theta(y_1)\theta(y_2)\omega^6F^2(\omega)e^{-\mathrm{i}\frac{\omega}{\varepsilon}\tau}\rho(z)\\&\times\hat{G}_0\Big(\frac{\omega}{\varepsilon},x_3,y_1\Big)\overline{\hat{G}}_0\Big(\frac{\omega}{\varepsilon},x_1,z\Big)\overline{\hat{G}}_0\Big(\frac{\omega}{\varepsilon},z,y_1\Big)\overline{\hat{G}}_0\Big(\frac{\omega}{\varepsilon},x_3,y_2\Big)\\&\times\hat{G}_0\Big(\frac{\omega}{\varepsilon},x_2,z_r\Big)\hat{G}_0\Big(\frac{\omega}{\varepsilon},z_r,z\Big)\hat{G}_0\Big(\frac{\omega}{\varepsilon},z,y_2\Big)\\=&\frac{\sigma_r\sigma_s^2 l_s^3 l_r^3}{2^{15}\pi^8\varepsilon^5 c_0^6}\int dy_1dy_2d\omega dz\theta(y_1)\theta(y_2)\omega^6F^2(\omega)e^{\mathrm{i}\frac{\omega}{\varepsilon}\mathcal{T}}\rho(z)\\&\times\frac{1}{|x_3-y_1||x_1-z||z-y_1||x_3-y_2||x_2-z_r||z_r-z||z-y_2|},\end{aligned}\tag{5.19}$$

where the phase term is given by,

$$\omega\mathcal{T}=\omega\Big[\tau(x_3,y_1)-\tau(x_1,z)-\tau(z,y_1)-\tau(x_3,y_2)+\tau(x_2,z_r)+\tau(z_r,z)+\tau(z,y_2)-\tau\Big]. \quad (5.20)$$

By the stationary phase method, the dominant contribution of (5.19) comes from the stationary points of the phase which satisfy

$$\partial_\omega(\omega\mathcal{T})=0, \quad \nabla_{y_1}(\omega\mathcal{T})=0, \quad \nabla_{y_2}(\omega\mathcal{T})=0, \quad \nabla_z(\omega\mathcal{T})=0.$$

This implies that

$$\tau=\tau(x_3,y_1)-\tau(x_1,z)-\tau(z,y_1)-\tau(x_3,y_2)+\tau(x_2,z_r)+\tau(z_r,z)+\tau(z,y_2), \quad (5.21)$$

$$\nabla_{y_1}\tau(x_3,y_1)=\nabla_{y_1}\tau(z,y_1), \qquad \nabla_{y_2}\tau(x_3,y_2)=\nabla_{y_2}\tau(z,y_2), \quad (5.22)$$

$$\nabla_z\tau(z_r,z)+\nabla_z\tau(z,y_2)=\nabla_z\tau(x_1,z)+\nabla_z\tau(z,y_1). \quad (5.23)$$

(5.22) implies the condition that $y_1,y_2\to x_3\to z$. And (5.23) implies the condition that $z\to x_1\to z_r$. Combining (5.21), we could conclude that (5.19) has a singular component at $\tau=\tau(x_1,z_r)+\tau(x_2,z_r)$. Similarly, we could analyze the other terms in the part with leading order $\mathcal{O}(\sigma_r\sigma_s^2)$. In fact, for some terms, we could get that they have a singular component at $\tau=-(\tau(x_1,z_r)+\tau(x_2,z_r))$. Thus we could conclude that in this case the second-order coda cross correlation has singular components of order $\mathcal{O}(\sigma_r\sigma_s^2)$ at $\tau=\pm(\tau(x_1,z_r)+\tau(x_2,z_r))$. □

Actually, the result given by Theorem 5.3 is expected. Since the second-order coda cross correlation mainly characterizes the influence of the scattering medium. Under the space configuration in Fig. 9, the scattering medium on the right end could be regarded as a secondary source due to reflection. Then a daylight configuration is constructed by this secondary source, together with the sensors and reflector. As we have shown in Theorem 2.3, the first-order cross correlation has singular components at $\pm(\tau(x_1,z_r)+\tau(x_2,z_r))$ under the daylight configuration. So the result for the second-order coda cross correlation just agrees with the first-order case.

6 Applications of the second-order cross correlation in the passive imaging

Until now we have analyzed the second-order cross correlation and the second-order coda cross correlation in homogeneous medium and scattering medium, respectively. In this section, we discuss how we could apply these properties to the passive imaging. Specially, we give two imaging methods in this section.

6.1 Imaging based on the imaging function

This imaging method has been discussed deeply in [14–16] with the first-order cross correlation, and it can be directly generalized to the second-order case. Firstly we consider the imaging problem in the homogeneous medium. According to Theorem 4.2, the second-order cross correlation $C^{(2)}$ has two types of singular components. One type is of order $\mathcal{O}(1)$, appearing at $\tau=\pm\tau(x_1,x_2)$. Another type is of order $\mathcal{O}(\sigma_r)$, appearing at $\tau=\tau(x_2,z_r)-\tau(x_1,z_r)$. To eliminate the influence of the direct waves, here we consider the following difference of the second-order cross correlation,

$$\Delta C^{(2)}(\tau,x_1,x_2)=C^{(2)}(\tau,x_1,x_2)-C_0^{(2)}(\tau,x_1,x_2). \tag{6.1}$$

Here $C^{(2)}$ the second-order cross correlation with a point reflector, which is given by (4.10). And $C_0^{(2)}$ is the second-order cross correlation without reflectors, which is given by (4.2). With this difference, we could construct the imaging function as follows,

$$I^H(z_s)=\Delta C^{(2)}(\tau(x_2,z_s)-\tau(x_1,z_s),x_1,x_2), \tag{6.2}$$

where z_s is the searching point in the searching area. By Theorem 4.2, we could expect that function (6.2) should have a local extreme point at z_r.

For the scattering medium case, we could do the similar work for imaging. Again to eliminate the influence of the direct waves, we consider the following difference of the second-order coda cross correlation,

$$\Delta C_{coda}^{(2)}(\tau,x_1,x_2)=C_{coda}^{(2)}(\tau,x_1,x_2)-C_{coda,0}^{(2)}(\tau,x_1,x_2). \tag{6.3}$$

Here $C_{coda}^{(2)}$ is the second-order coda cross correlation with a point reflector, which is given by (5.18). And $C_{coda,0}^{(2)}$ is the second-order coda cross correlation without reflectors, which is given by (5.10). With this difference, we could construct the imaging function as follows,

$$I^S(z_s)=\Delta C_{coda}^{(2)sym}(\tau(x_2,z_s)+\tau(x_1,z_s),x_1,x_2), \tag{6.4}$$

where

$$\Delta C_{coda}^{(2)sym}(\tau,x_1,x_2)=\Big[\Delta C_{coda}^{(2)}(\tau,x_1,x_2)+\Delta C_{coda}^{(2)}(-\tau,x_1,x_2)\Big]1_{(0,\infty)}(\tau). \tag{6.5}$$

Again z_s in (6.4) is the searching point in the searching area. By Theorem 5.3, we could expect that the expectation of (6.4) should have a local extreme point at z_r.

Remark 6.1. For the imaging function (6.4), due to the stochastic scattering background medium, the quantity I^S is a stochastic quantity. So if we want to do the imaging work with this imaging function, we need take the signal-to-noise rate (SNR) into consideration. In the statistic sense, a larger SNR implies a more stable imaging result. Actually, the authors have given a detailed SNR analysis in [27]. Indeed, the SNR of the imaging function (6.4) is of order $\mathcal{O}(1)$ (with respect to σ_s). On the other hand, as stated in [16], we could also use the first-order coda cross correlation to construct the corresponding imaging function in this scattering medium case. However, for the imaging function with the first-order coda cross correlation, authors in [16] show that the corresponding SNR is only of order $\mathcal{O}(\sigma_s)$, which is much smaller than that with the second-order coda cross correlation.

The following example gives an illustration of passive imaging in the homogeneous medium. In this example, the function-based method proposed above is used.

Example 6.1. We take the same setting as that in Example 4.2. For this example, we use the imaging function (6.2) to find the target point reflector which is located at $z_r = (-0.25,0)$. Fig. 10 illustrates the imaging result with this imaging function in the searching area $[-0.5,0.15]\times[-0.5,0.5]$. The black square in the figure is the exact location of the target reflector. Similarly as the result for the first-order case [15], compared to the cross-range resolution, here the imaging result does not have a good range resolution.

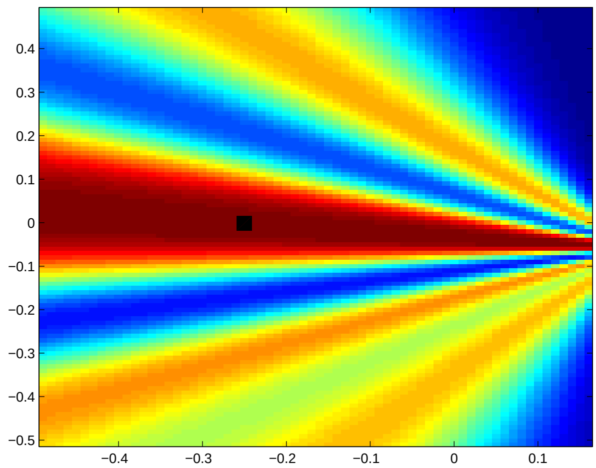

Figure 10: The imaging result with the imaging function (6.2) at the searching area $[-0.5,0.15]\times[-0.5,0.5]$. The black square in the figure corresponds to the exact location of the target reflector.

6.2 Imaging based on the geometric property

In the previous subsection, we have introduced an imaging method based on the imaging function. However, as shown in Example 6.1, when we use imaging function (6.2) in the homogeneous case, due to the backlight configuration, we could not get a result with a good range resolution. Indeed, Garnier and Papanicolaou have given a detailed resolution analysis on this backlight imaging function in [15]. To try to overcome this shortcoming, in this subsection, we will introduce an imaging method based on the geometric property.

According to Theorem 4.2, we know that the second-order cross correlation has a singular component of order $\mathcal{O}(\sigma_r)$ at $\tau=\tau(x_2,z_r)-\tau(x_1,z_r)$. Multiplying the background velocity c_0 on both sides, we get,

$$c_0\tau=|x_2-z_r|-|x_1-z_r|. \tag{6.6}$$

In the geometric sense, the equation above means that the location of the point reflector z_r should lie on the half branch of the hyperbola near x_1, of which the focal points are x_1,x_2, and the difference of the distances to the focal points is $c_0\tau$, or sometimes degenerates to a line. On the other hand, by the observation of the wave signals at the sensors, we could calculate the second-order cross correlation between x_1,x_2. Then we could obtain the location τ of the singular component of order $\mathcal{O}(\sigma_r)$. Thus, we could plot the half branch of the hyperbola with the focal points x_1,x_2 and the distance difference $c_0\tau$. Similarly, if we use another set of sensors, we could plot another hyperbola. With the intersection of these two hyperbolas, we could find the location of the target point reflector. Further more, if more sets of sensors could be given, we could get more hyperbolas. Then the intersection area of all the hyperbolas could be a good estimation of the location of the target reflector. The following example shows how this imaging method works.

Example 6.2. For this example, we use the geometric-based imaging method to find the target reflector under the same setting as that in Example 4.2. Firstly, we set the sensors at $x_1=(0.2,-0.1),x_2=(0.2,0)$, and set the auxiliary sensor at $x_3=(-0.1,0)$. By the wave signals collected from the set of sensors, we get the second-order cross correlation of order $\mathcal{O}(\sigma_r)$ as show in the left part of Fig. 11, which has a singular component near $\tau=-0.2$. Then, we set the sensors at $x_1=(-0.15,0),x_2=(-0.05,0)$, and set the auxiliary sensor at $x_3=(-0.2,0)$. Again by the wave signals collected from the set of sensors, we get the second-order cross correlation of order $\mathcal{O}(\sigma_r)$ as shown in the right part of Fig. 11, which has a singular component near $\tau=2$s. With the two results of second-order cross correlations, we could plot two hyperbolas. Fig. 12 illustrates these two hyperbolas. The black square in the figure is the exact location of the target reflector. We could see that the intersection of these two hyperbolas is very close to the black square, which means that this geometric-based imaging method is efficient.

7 Conclusion

In this paper, following Garnier's and Papanicolaou's works in [14–16], we have introduced the second-order cross correlation and the second-order coda cross correlation in homogeneous medium and scattering medium, respectively. We have proven that both two cross correlations are statistically stable quantities. More precisely, the convergent rate of the second-order cross correlation is of order $\mathcal{O}(T^{-1+\alpha})$, which is faster, if $0<\alpha<1/2$, than that of the first-order one, of which the convergent rate is only of order $\mathcal{O}(T^{-1/2})$.

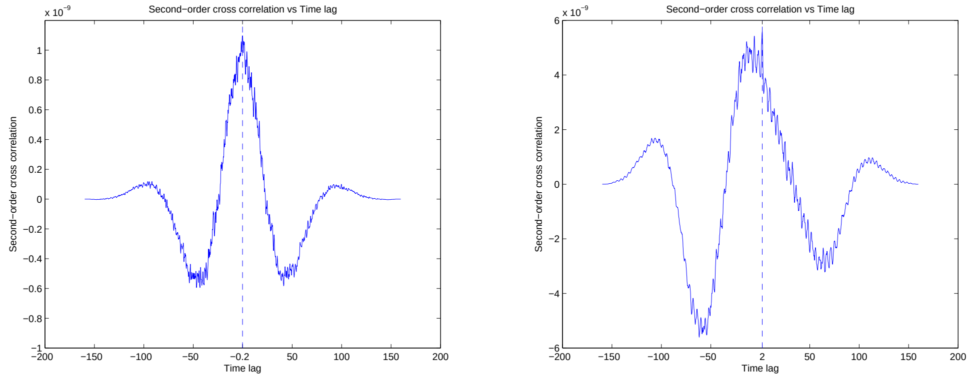

Figure 11: The two second-order cross correlations of order $\mathcal{O}(\sigma_r)$. The left one corresponds to the sensors located at $x_1=(0.2,-0.1), x_2=(0.2,0), x_3=(-0.1,0)$. The right one corresponds to the sensors located at $x_1=(-0.15,0), x_2=(-0.05,0), x_3=(-0.2,0)$.

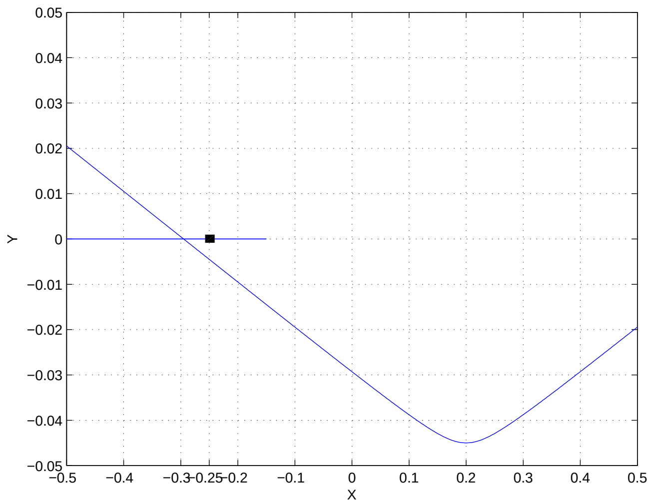

Figure 12: The two hyperbolas got from the information of the second-order cross correlations. The black square is the exact location of the target reflector.

Moreover, with the stationary phase method, we have analyzed the behavior of the second-order cross correlation in the homogeneous medium, and the behavior of the second-order coda cross correlation in the scattering medium. Indeed, we get that both these two cross correlations have a singular component of leading order at $\tau=\pm\tau(x_1,x_2)$. Specially, under the backlight configuration, the second-order cross correlation has a singular component of order $\mathcal{O}(\sigma_r)$ at $\tau=\tau(x_2,z_r)-\tau(x_1,z_r)$, and the second-order coda cross correlation has singular components of order $\mathcal{O}(\sigma_r\sigma_s^2)$ at $\tau=\pm(\tau(x_1,z_r)+\tau(x_2,z_r))$. These behaviors agree quite well with that of the first-order cross correlations.

Finally, with the properties of both the second-order cross correlation and the second-order coda cross correlation, we have proposed two imaging methods. The first method is based on the imaging function. Specially, for the scattering medium, due to the fact that the SNR of the imaging function constructed with the second-order coda cross correlation is of order $\mathcal{O}(1)$ while the SNR of the imaging function constructed with the first-order coda cross correlation is of order $\mathcal{O}(\sigma_s)$, we conclude that it may be more stable to image using the second-order coda cross correlation than using the first-order coda cross correlation in the scattering medium. The second method is based on the geometric property. Specially we have proposed an imaging method in the homogeneous medium to improve the bad range resolution in the first function-based method, which is caused by the backlight configuration. Indeed, by setting two or more different sets of sensors, corresponding hyperbolas could be plotted with the information of the sensor locations and the corresponding singular components of the second-order cross correlation. And the intersection area of the hyperbolas is just a possible estimation of the location of the target point reflector.

Acknowledgments

The authors are supported by the Ministry of Education of China & State Administration of Foreign Experts Affairs of China under the 111 project grant (B08018), the Key Project National Science Foundation of China (91130004) and Natural Science Foundation of China (11171077, 11331004). The authors also thank Qifu Chen, Cheng Hua, Xiaoming Wang and Huaxiong Huang for their valuable discussions.

References

[1] H. Ammari, E. Bretin, J. Garnier and A. Wahab, Noise source localization in an attenuating medium, SIAM J. Appl. Math., 72(2012), 317–336.

[2] H. Ammari, J. Garnier and W. Jing, Passive array correlation-based imaging in a random waveguide, Multiscale Model. Simul., 11(2013), no. 2, 656–681.

[3] H. Ammari, J. Garnier, W. Jing, H. Kang, M. Lim, K. Solna and H. Wang, Mathematical and statistical methods for multistatic imaging, Lecture Notes in Mathematics, 2098, Springer, Cham, 2013.

[4] H. Ammari, J. Garnier, and V. Jugnon, Detection, reconstruction, and characterization algorithms from noisy data in multistatic wave imaging, Discrete Contin. Dyn. Syst. Ser. S8(2015), no. 3, 389–417.

[5] C. Bardos, J. Garnier and G. Papanicolaou, Identification of Green's functions singularities by cross correlation of noisy signals, Inverse Problems, 24(2008), 015011.

[6] N. Bleistein, J. K. Cohen and J. W. Stockwell Jr, Mathematics of multidimensional seismic imaging, migration, and inversion, Springer Verlag, New York, 2001.

[7] L. Borcea, G. Papanicolaou and C. Tsogka, Adaptive interferometric imaging in clutter and optimal illumination, Inverse Problems, 22(2006), 1405–1436.

[8] L. Borcea, G. Papanicolaou and C. Tsogka, Optimal illumination and waveform design for imaging in random media, J. Acoust. Soc. Am., 122(2007), 3507–3518.

[9] M. Born and E. Wolf, Principles of optics, Cambridge University Press, Cambridge, 1999.

[10] F. Brenguier, N. M. Shapiro, M. Campillo, V. Ferrazzini, Z. Duputel, O. Coutant and A. Nercessian, Towards forecasting volcanic eruptions using seismic noise, Nature Geoscience, 1(2008), 126–130.

[11] F. Brenguier, N. M. Shapiro, M. Campillo, A. Nercessian and V. Ferrazzini, 3-D surface wave tomography of the Piton de la Fournaise volcano using seismic noise correlations, Geophys. Res. Lett., 34(2007), L02305.

[12] A. Curtis, P. Gerstoft, H. Sato, R. Snieder and K. Wapenaar, Seismic interferometry-turning noise into signal, The Leading Edge, 25(2006), 1082–1092.

[13] T. L. Duvall Jr, S. M. Jefferies, J. W. Harvey and M. A. Pomerantz, Time-distance helioseismology, Nature, 362(1993), 430–432.

[14] J. Garnier and G. Papanicolaou, Passive sensor imaging using cross correlations of noisy signals in a scattering medium, SIAM J. Imaging Sciences, 2(2009), 396–437.

[15] J. Garnier and G. Papanicolaou, Resolution analysis for imaging with noise, Inverse Problem, 26(2010), 074001.

[16] J. Garnier and G. Papanicolaou, Resolution enhancement from scattering in passive sensor imaging with cross correlations, Inverse Problems and Imaging, 8(2014), 645–683.

[17] L. Isserlis, On a formula for the product-moment coefficient of any order of a normal frequency distribution in any number of variables, Biometrika, 12(1918), 134–139.

[18] C. Lanczos, Linear Differential Operators, Van Nostrand, London, 1961.

[19] E. Larose, L. Margerin, A. Derode, B. Van Tiggelen, M. Campillo, N. Shapiro, A. Paul, L. Stehly and M. Tanter, Correlation of random wave fields: an interdisciplinary review, Geophysics, 71(2006), SI11–SI21.

[20] C. Qi, Q. Chen and Y. Chen, A new method for seismic imaging from ambient seismic noise, Progress in Geophysics, 22(2007), 771–777.

[21] J. Rickett and J. Claerbout, Acoustic daylight imaging via spectral factorization: Helioseismology and reservoir monitoring, The Leading Edge, 18(1999), 957–960.

[22] G. T. Schuster, J. Yu, J. Sheng and J. Rickett, Interferometric daylight seismic imaging, Geophysical Journal International, 157(2004), 832–852.

[23] N. M. Shapiro, M. Campillo, L. Stehly and M. H. Ritzwoller, High-resolution surface wave tomography from ambient noise, Science, 307(2005), 1615–1618.

[24] R. Snieder, Extrating the Green's function of attenuating heterogeneous acoustic media from uncorrelated waves, J. Acoust. Soc. Amer., 121(2007), 2637–2643.

[25] L. Stehly, M. Campillo and N. M. Shapiro, A study of the seismic noise from its long-range correlation properties, Geophys. Res. Lett., 111(2006), B10306.

[26] Y. Colin De Verdière, Semiclassical analysis and passive imaging, Nonlinearity, 22(2009), R45–R75.

[27] L. Wang, Theorical analysis and numerical methods for a series of problems of wave equatinos with different source terms, Ph.D thesis, Fudan University, 2015.

[28] K. Wapenaar and J. Fokkema, Green's function representations for seismic interferometry, Geophysics, 71(2006), SI33–SI46.

[29] R. Weaver and O. I. Lobkis, Ultrasonics without a source: Thermal fluctuarion crrelations at MHz frequencies, Phys. Rev. Lett., 87(2001), 134301.