A Numerical Study of Complex Reconstruction in
Inverse Elastic Scattering

Guanghui Hu, Jingzhi Li, Hongyu Liu and Qi Wang

1 Beijing Computational Science Research Center, Beijing 100094, P.R. China.
2 Department of Mathematics, Southern University of Science and Technology, Shenzhen 518055, P.R. China.
3 Department of Mathematics, Hong Kong Baptist University, Kowloon Tong, Hong Kong.
4 Department of Computing Sciences, School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, 710049, P.R. China.

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Abstract. The purpose of this paper is to numerically realize the inverse scattering scheme proposed in [19] of reconstructing complex elastic objects by a single far-field measurement. The unknown elastic scatterers might consist of both rigid bodies and traction-free cavities with components of multiscale sizes presented simultaneously. We conduct extensive numerical experiments to show the effectiveness and efficiency of the imaging scheme proposed in [19]. Moreover, we develop a two-stage technique, which can significantly speed up the reconstruction to yield a fast imaging scheme.

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1 Introduction

This work concerns the numerical realization of an imaging scheme proposed in [19] for reconstructing complex elastic scatterers embedded in a homogeneous isotropic background medium occupying $\mathbb{R}^3$. Let $\lambda$ and $\mu$ be two constants such that $\mu > 0$ and $3\lambda + 2\mu > 0$. $\lambda$ and $\mu$ are the Lamé constants that constitute the parameterization of the background elastic material. Throughout, we assume that the density of the background elastic medium is normalized to be 1. Let $D \subset \mathbb{R}^3$ be a bounded domain with a $C^2$ boundary $\partial D$ and a connected complement $\mathbb{R}^3 \setminus \overline{D}$. $D$ denotes the inhomogeneous elastic body

*Corresponding author. Email addresses: hu@csrc.ac.cn (G. Hu), li.jz@sustc.edu.cn (J. Li), hongyu.liu@gmail.com (H. Liu), qi.wang.xjtumath@gmail.com (Q. Wang)
that we intend to recover by using elastic wave measurements made away from it. In
what follows, $D$ is referred to as a scatterer. The detecting elastic field is taken to be the
time-harmonic plane wave of the form

$$u^{in}(x) = u^{in}(x; d, d^\perp, \alpha, \beta, \omega) = \alpha d e^{ik_p x \cdot d} + \beta d^\perp e^{ik_s x \cdot d}, \quad \alpha, \beta \in \mathbb{C}, \quad (1.1)$$

where $d \in S^2 := \{ x \in \mathbb{R}^3 : |x| = 1 \}$ is the incident direction; the vector $d^\perp \in S^2$ satisfying $d^\perp \cdot d = 0$ denotes the polarization direction; and $k_s := \omega / \sqrt{\mu}$, $k_p := \omega / \sqrt{\lambda + 2\mu}$ denote the shear and compressional wave numbers, respectively. Let $u^{sc}(x) \in \mathbb{C}^3, x \in \mathbb{R}^3 \setminus \overline{D}$ denote the perturbed/scattered elastic displacement field caused by the elastic scatterer and $u := u^{in} + u^{sc}$ denote the total field. The propagation of the elastic field is governed by the following reduced Navier equation (or Lamé system)

$$(\triangle^* + \omega^2) u = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{D}, \quad \triangle^* := \mu \triangle + (\lambda + \mu) \text{grad div.} \quad (1.2)$$

In order to complete the description of the direct elastic scattering problem, we next prescribe the physically meaningful boundary conditions satisfied by the elastic field on $\partial D$ and at the infinity.

Define the infinitesimal strain tensor by

$$\varepsilon(u) := \frac{1}{2} \left( \nabla u + \nabla u^T \right) \in \mathbb{C}^{3 \times 3}, \quad (1.3)$$

where $\nabla u$ and $\nabla u^T$ stand for the Jacobian matrix of $u$ and its adjoint, respectively. By Hooke’s law the Cauchy stress tensor relates to the strain tensor via the identity

$$\sigma(u) = \lambda (\text{div} u) I + 2\mu \varepsilon(u) \in \mathbb{C}^{3 \times 3}, \quad (1.4)$$

where $I$ denotes the $3 \times 3$ identity matrix. The surface traction (or the stress operator) on $\partial D$ is defined as

$$T u = T_v := \nu \cdot \sigma(u) = (2\mu \nu \cdot \text{grad} + \lambda \nu \text{div} + \mu \nu \times \text{curl}) u, \quad (1.5)$$

where $\nu$ denotes the unit normal vector to $\partial D$ pointing into $\mathbb{R}^3 \setminus D$. We also define $R u := u$ in the following. If $D$ is a cavity, then one has the traction-free boundary condition $T u = 0$ on $\partial D$; and if $D$ is a rigid body, then one has $R u = 0$ on $\partial D$.

Decomposing the incident wave $u^{in}$ in (1.1), we denote by $u^{in}_p := d e^{ik_p x \cdot d}$ the (normalized) plane pressure wave, and $u^{in}_s := d^\perp e^{ik_s x \cdot d}$ the (normalized) plane shear wave. By Hodge decomposition, the scattered field $u^{sc}$ can be decomposed into

$$u^{sc} := u^{sc}_p + u^{sc}_s, \quad u^{sc}_p := -\frac{1}{k_p^2} \text{grad div} u^{sc}, \quad u^{sc}_s := \frac{1}{k_s^2} \text{curl curl} u^{sc},$$