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A Novel Technique for Constructing Difference Schemes for Systems of Singularly Perturbed Equations

Po-Wen Hsieh[1], Yin-Tzer Shih[2], Suh-Yuh Yang[3,*] and Cheng-Shu You[3]

[1] *Department of Applied Mathematics, Chung Yuan Christian University, Jhongli District, Taoyuan City 32023, Taiwan.*
[2] *Department of Applied Mathematics, National Chung Hsing University, Taichung 40227, Taiwan.*
[3] *Department of Mathematics, National Central University, Jhongli District, Taoyuan City 32001, Taiwan.*

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Abstract. In this paper, we propose a novel and simple technique to construct effective difference schemes for solving systems of singularly perturbed convection-diffusion-reaction equations, whose solutions may display boundary or interior layers. We illustrate the technique by taking the Il'in-Allen-Southwell scheme for 1-D scalar equations as a basis to derive a formally second-order scheme for 1-D coupled systems and then extend the scheme to 2-D case by employing an alternating direction approach. Numerical examples are given to demonstrate the high performance of the obtained scheme on uniform meshes as well as piecewise-uniform Shishkin meshes.

AMS subject classifications: 65N06, 65N12, 76M20

Key words: System of singularly perturbed equations, system of viscous Burgers' equations, boundary layer, interior layer, Il'in-Allen-Southwell scheme, formally second-order scheme.

1 Introduction

The aim of this paper is to demonstrate a novel and simple technique to construct effective difference schemes for solving systems of singularly perturbed convection-diffusion-reaction (CDR) equations, whose solutions may display boundary or interior layers. Let $\Omega\subset\mathbb{R}^2$ be an open, bounded and convex polygonal domain with boundary $\partial\Omega$. We consider the Dirichlet boundary value problem for a system of two linear CDR equations:

$$\begin{cases} -E\Delta\boldsymbol{u}-A\boldsymbol{u}_x-B\boldsymbol{u}_y+C\boldsymbol{u}=\boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{u}=\boldsymbol{g} & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

*Corresponding author. *Email addresses:* pwhsieh0209@gmail.com (P.-W. Hsieh), yintzer_shih@nchu.edu.tw (Y.-T. Shih), syyang@math.ncu.edu.tw (S.-Y. Yang), csdyou@gmail.com (C.-S. You)

where $\boldsymbol{u}=(u_1,u_2)^\top$ is the physical quantity of interest; $E=\mathrm{diag}\{\varepsilon_1,\varepsilon_2\}$ is a given constant matrix of diffusivities with $0<\varepsilon_i\le 1$ and we are particularly interested in the case of $\varepsilon_1=\varepsilon_2\ll 1$; $A=(a_{ij})_{2\times 2}$ and $B=(b_{ij})_{2\times 2}$ are the given convection coefficient matrices and $C=(c_{ij})_{2\times 2}$ is the reaction coefficient matrix; $\boldsymbol{f}=(f_1,f_2)^\top$ is a given source term and $\boldsymbol{g}=(g_1,g_2)^\top$ is a prescribed boundary data. Throughout this paper, we always assume that the coefficient matrices A,B,C, with $a_{ii}\neq 0$ and $b_{ii}\neq 0$ in Ω for $i=1,2$, and $\boldsymbol{f},\boldsymbol{g}$ are sufficiently smooth such that problem (1.1) is well posed.

Compared with the scalar singularly perturbed CDR equation, system (1.1) can model more complicated physical phenomena, such as the turbulent interaction of waves and currents [15], the diffusion processes in the presence of chemical reactions [14], the optimal control and certain resistance-capacitor electrical circuits [6], and the magnetohydrodynamic duct flow problems [2, 3], etc. Similar to the scalar equation, as one of the diffusivities ε_i is small enough than the module of the corresponding convection or reaction coefficients, the solution component u_i of (1.1) may display boundary or interior layers. These layers are narrow regions where the solution component changes rapidly and it is often difficult to resolve numerically the high gradients near the layer regions. Therefore, the study of singularly perturbed problems has been the focus of intense research for quite some time. However, most numerical methods for such problems are lacking in either stability or accuracy (cf. [7, 12]). For example, the central difference scheme performs very poorly since large spurious oscillations appear.

Aiming to overcome the difficulties caused by high gradients of solution of system of singularly perturbed CDR equations, most difference schemes are essentially of the upwind type and thus have only first-order accuracy. In [8], O'Riordan *et al.* proposed a difference scheme for 1-D case which is consisting of simple upwinding with an appropriate piecewise-uniform Shishkin mesh. They showed the first-order convergence when A is a strictly diagonally dominant M-matrix and $C=\mathbf{0}$. In [9], with proper hypotheses placed on the coupling matrices A and C, a similar scheme was designed for more general 1-D systems. Also, they obtained the first-order approximations by using a Jacobi-type iteration [10]. For 2-D systems of singularly perturbed CDR equations, we proposed in [3] a compact difference scheme with accuracy of $\mathcal{O}(\varepsilon^2(h+k)+\varepsilon(h^2+k^2)+(h^3+k^3))$ when A and B are symmetry matrices with zero diagonal entries and $C=\mathbf{0}$. To the best of our knowledge, it seems not easy to construct workable difference schemes, rather than the upwind-type schemes, for systems of singularly perturbed CDR equations.

In this paper, we will propose a novel and simple technique to construct effective difference schemes for solving 2-D systems of singularly perturbed CDR equations, whose solutions may display boundary or interior layers. We will illustrate the technique by taking the Il'in-Allen-Southwell scheme [11, 12] as a basis and combining with a novel treatment for the diffusion terms to derive a three-point difference scheme for the 1-D counterpart of system (1.1). We then extend the scheme to the 2-D coupled system (1.1) by employing an alternating direction approach [3, 16]. We remark that the Il'in-Allen-Southwell scheme is a formally second-order difference scheme for scalar CDR equa-