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A Dimensional Splitting of ETD Schemes for Reaction-Diffusion Systems

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Abstract. Novel dimensional splitting techniques are developed for ETD Schemes which are second-order convergent and highly efficient. By using the ETD-Crank-Nicolson scheme we show that the proposed techniques can reduce the computational time for nonlinear reaction-diffusion systems by up to 70%. Numerical tests are performed to empirically validate the superior performance of the splitting methods.

AMS subject classifications: 65M12, 65M15, 65M20

Key words: Exponential time differencing, dimensional splitting, reaction diffusion equations.

1 Introduction

Reaction-diffusion systems are mathematical models with a wide variety of applications, *e.g.* [10, 11, 13, 15, 16]. Reaction-diffusion systems can be reduced to systems of stiff ordinary differential equations by employing a method-of-lines discretization. Numerous time-discretization methods for solving the resulting stiff ODE systems have been reported in the literature. Among these are linearly implicit methods [8, 19, 20], semi-implicit methods [17], and projection methods [18]. More recently Exponential Time Differencing (ETD) schemes have been developed [1, 12, 15], which make use of a single step representation of the evolutionary dynamics followed by an appropriate discretization of the exponentials that arise. Variants of the proposed ETD scheme have been developed over the years which adopt different approximations to the integral resulting from the nonlinear reaction term [2, 3, 5, 6, 12].

For multidimensional systems, the discretized diffusion matrices are often largely sparse with wide bands, which tend to slow down direct solvers during evolution. In [7] an ETD-LOD scheme was developed to reduce the storage requirements for solving

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higher dimensional problems and speed up the evolution through extrapolation of a first order ETD scheme and a simple type of locally one-dimensional (LOD) splitting to achieve second order accuracy. In the present article we are concerned with speed-up through different types of locally one-dimensional-splitting and without using extrapolation, instead, directly working with a second order ETD scheme. In the first approach we break up the PDE into sub-problems and apply a Strang composition of the sub-solution operators to recover the solution. The second method uses an integrating factor substitution to achieve a natural splitting of the PDE along its spatial dimensions. We apply the technique to split the second order ETD-CN [5] and examine its performance in discretizing several two dimensional problems.

2 ETD-schemes

Consider the following semilinear parabolic initial-boundary value problem:

$$\begin{aligned} &u_t+\mathcal{A}u=F(t,u) \quad \text{in } \Omega, \quad t\in(0,T], \\ &u(\cdot,0)=u_0 \qquad\quad \text{in } \Omega, \\ &\mathcal{L}u(\cdot,t)=0 \qquad\quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where Ω is a bounded domain in $\Re^d$ with Lipschitz continuous boundary, $\mathcal{A}$ represents a uniformly elliptic operator and F is a sufficiently smooth, nonlinear reaction term. The operator $\mathcal{A}$ is usually of the form,

$$\mathcal{A}:=-\sum_{j,k=1}^{d}\frac{\partial}{\partial x_j}\left(a_{j,k}(x)\frac{\partial}{\partial x_k}\right)+\sum_{j=1}^{d}b_j(x)\frac{\partial}{\partial x_j}+b_0(x),$$

where the coefficients $a_{j,k}$ and b_j are sufficiently smooth functions on $\overline{\Omega}$, $a_{j,k}=a_{k,j}$, $b_0\geq 0$, and for some $c_0>0$

$$\sum_{j,k=1}^{d}a_{j,k}(\cdot)\xi_j\xi_k\geq c_0|\xi|^2, \quad \text{on } \overline{\Omega}, \quad \text{for all } \xi\in\mathbb{R}^d.$$

The boundary operator is $\mathcal{L}u=u$ (Dirichlet) or $\frac{\partial u}{\partial n}$ (Neumann) boundary conditions. From the Duhamel principle, the solution for (2.1) can be expressed in terms of the recurrence relation

$$u(t_{n+1})=e^{-\mathcal{A}k}u(t_n)+\int_{t_n}^{t_{n+1}}e^{-\mathcal{A}(t_{n+1}-s)}F(s,u(s))ds,$$

which can be simplified further to

$$u(t_{n+1})=e^{-\mathcal{A}k}u(t_n)+\int_0^k e^{-\mathcal{A}(k-\tau)}F(t_n+\tau,u(t_n+\tau))d\tau, \tag{2.2}$$