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Application of Lie Algebra in Constructing Volume-Preserving Algorithms for Charged Particles Dynamics

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Abstract. Volume-preserving algorithms (VPAs) for the charged particles dynamics is preferred because of their long-term accuracy and conservativeness for phase space volume. Lie algebra and the Baker-Campbell-Hausdorff (BCH) formula can be used as a fundamental theoretical tool to construct VPAs. Using the Lie algebra structure of vector fields, we split the volume-preserving vector field for charged particle dynamics into three volume-preserving parts (sub-algebras), and find the corresponding Lie subgroups. Proper combinations of these subgroups generate volume preserving, second order approximations of the original solution group, and thus second order VPAs. The developed VPAs also show their significant effectiveness in conserving phase-space volume exactly and bounding energy error over long-term simulations.

AMS subject classifications: 17B81, 22E70, 70H33, 70H40

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1 Introduction

Dynamics of non-relativistic and relativistic charged particles arises commonly in plasma physics, accelerator physics, space physics, and many other subfields of physics [1–4, 8,

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10–12, 16–18]. Because charged particle dynamics in a general electromagnetic field preserves phase space volume, it is desirable to construct corresponding volume-preserving algorithms for numerical integrations [5, 15]. Volume-preserving algorithms have been successfully constructed and their superior advantages in terms of long-term accuracy and conservative properties have been demonstrated [9, 14, 19]. It is well-known that vector fields of a dynamical system on a given manifold have the structure of Lie algebra. An important method to construct VPAs is based on the splitting technique applied to the Lie algebra [6,7]. In this method, the original vector field for the dynamics in phase space is split into several parts, such that for each part the system can be solved exactly or a VPA can be easily found [13]. Combining all the sub-algorithms in a proper way, we can obtain a desired VPA for the original system. The procedure to construct VPAs can be executed in three steps:

1) To split the original vector field into several sub-vector fields;

2) To find corresponding Lie groups of the Lie algebra generated by the each sub-vector field;

3) To combine Lie subgroups to obtain a desired VPA for the original system.

Many different VPAs can be constructed following the above three steps, because there are different degrees of freedom in every step. In the first step, a given vector fields can be split in different ways. In the second step, we can choose either the exact solution or an approximate solution which is volume-preserving. In the third step, different composition methods can be used to generate algorithms with different orders and symmetry properties. The order of a VPA is determined by order of the approximation to the original one-parameter Lie group according to the BCH formula.

In this paper, we construct VPAs for non-relativistic and relativistic dynamics of charged particles using the method of Lie algebra by splitting the vector field in phase space into three parts: the force-free part, the electrical part, and the magnetic part. For each sub-vector field the corresponding Lie subgroup can be solved exactly or approximately by the Cayley transformation. Desired VPAs can be constructed symmetrically using the Strang composition method. Applying the developed VPAs to calculate the ware-pinch effect in tokamak, we demonstrate the long-term accuracy and significance of the VPA compared with RK4 method.

The paper is organized as follows. Some basic facts about splitting methods in the view of Lie algebra and Lie group are introduced in Section 2. In Section 3, we apply the theory of Lie algebra to construct VPAs for non-relativistic charged particles dynamics, and VPAs for relativistic dynamics are given in Section 4. Numerical experiments are also provided to demonstrate the effectiveness of VPAs compared with RK4 in Section 5.

2 Lie algebra and its application in splitting methods

For an initial value problem