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A New Weak Galerkin Finite Element Scheme for the Brinkman Model

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Abstract. The Brinkman model describes flow of fluid in complex porous media with a high-contrast permeability coefficient such that the flow is dominated by Darcy in some regions and by Stokes in others. A weak Galerkin (WG) finite element method for solving the Brinkman equations in two or three dimensional spaces by using polynomials is developed and analyzed. The WG method is designed by using the generalized functions and their weak derivatives which are defined as generalized distributions. The variational form we considered in this paper is based on two gradient operators which is different from the usual gradient-divergence operators for Brinkman equations. The WG method is highly flexible by allowing the use of discontinuous functions on arbitrary polygons or polyhedra with certain shape regularity. Optimal-order error estimates are established for the corresponding WG finite element solutions in various norms. Some computational results are presented to demonstrate the robustness, reliability, accuracy, and flexibility of the WG method for the Brinkman equations.

AMS subject classifications: 65N30, 65N15, 65N12, 74N20

Key words: Weak Galerkin finite element methods, weak gradient, Brinkman equation, polyhedral meshes.

1 Introduction

Incompressible viscous flows around and through porous arise in many fields, such as underground water hydrology, petroleum industry, automotive industry, biomedical engineering, and heat pipes modeling [11, 21, 22]. In these applications, the simple Darcy model is just suit for modeling slow flow problems [10, 16], thus can not describe cavity

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problems well. Furthermore, even though the Stokes-Darcy interface model is suit for the flow of a viscous fluid in porous and cavity media, it is not practical for the accuracy information about the number and locations of the interfaces between vugs and the porous matrix are not accessible. In 1949, H. C. Brinkman proposed a so called Brinkman model [3] for this transport phenomena in porous media. The unified equations in the Brinkman model represent flow of fluid in complex porous media with a permeability coefficient highly varying such that the flow is dominated by Stokes in some regions and by Darcy in others. Thus, comparing with the popular Stokes-Darcy interface model, the Brinkman model can describe both a Stokes and a Darcy flow without employing complex interface conditions.

Mathematically speaking the Brinkman model is a parameter dependent combination of the Darcy and Stokes models. The challenging aspects are mainly coming from the high variability in the PDE coefficients, that may take extremely large or small values, negatively affects the conditioning of the discrete problem which poses a substantial challenge for developing efficient and stable algorithm to suit for both the Stokes and Darcy simultaneously. In literature, a great deal of effort has been made in meeting this challenge by modifying either existing Stokes elements or Darcy elements to obtain new Brinkman stable elements. In [12], the numerical experiments show that for certain stable Stokes elements, such as Taylor-Hood, P_2-P_0, mini elements, will lead to non-convergent discretizations as the Brinkman equations become Darcy-dominated and similarly, for certain stable Darcy elements, such as Raviart-Thomas elements will lead to non-convergence as the Brinkman becomes Stokes-dominated. A large number of approaches have been proposed in the literature to address the numerical stability of various discretized methods. Among these work, in [4], [5] and [23], the authors introduce jumps penalization on the normal component of the velocity field or on the pressure field to stabilize the Crouzeix-Raviart-P_0 finite elements or P_1-P_0 finite elements, respectively. In [6], an augmented Lagrangian approach and a least squares stabilization are explored in order to use inf-sup compatible Taylor-Hood elements also in the Darcy case, while in [12] and [9] high order non-conforming elements are investigated.

In the present paper, we analyze the development of stable numerical methods for the Brinkman equations by using weak Galerkin finite element methods. It shows that the WG presents a natural and straightforward framework for constructing stable numerical algorithms for the Brinkman equations. Weak Galerkin (WG) refers to finite element techniques for partial differential equations in which differential operators are approximated by weak forms as distributions. Weak Galerkin methods were first introduced in [17] for the second order elliptic problem and was further developed in [13, 15, 18, 19, 24, 25] with other applications. The central idea of the WG method is to interpret the differential operators (e.g., gradient, Laplacian, Hessian, curl, divergence etc.) as distributions over a space of generalized functions and employ some proper stabilizations to enforce weak continuities for approximating functions. It has shown that the WG methods are efficient and robust by allowing the use of discontinuous approximating functions. The flexibility of discontinuous functions gives WG methods many advantages, such as high