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Parameter-Free Time Adaptivity Based on Energy Evolution for the Cahn-Hilliard Equation

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Abstract. It is known that large time-stepping method are useful for simulating phase field models. In this work, an adaptive time-stepping strategy is proposed based on numerical energy stability and equi-distribution principle. The main idea is to use the energy variation as an indicator to update the time step, so that the resulting algorithm is free of user-defined parameters, which is different from several existing approaches. Some numerical experiments are presented to illustrate the effectiveness of the algorithms.

AMS subject classifications: 65N10, 65N15, 35J25, 35Q99, 35R99, 65M12, 65M70

Key words: Adaptive time-stepping method, Cahn-Hilliard equation, Crank-Nicolson scheme, convex splitting method.

1 Introduction

This paper is concerned with the numerical method for the Cahn-Hilliard equation

$$\frac{\partial u}{\partial t}+\Delta(u-u^3+\kappa\Delta u)=0, \quad (x,t)\in\Omega\times(0,T], \tag{1.1a}$$

$$u(x,0)=u_0(x), \tag{1.1b}$$

where $\Omega=(0,L_1)\times(0,L_2)$ is a simple domain for the sake of simplicity, κ is a positive constant parameter and $u_0(x)$ is the initial data. Also for simplicity, the periodic boundary

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condition is imposed. The Cahn-Hilliard equation is first introduced in [1] to describe a continuum model for two mixture components separation and coarsening phenomena. In the model, u represents the concentration of one in the two components and the small parameter κ relates to the interfacial width.

There have been a large number of research work dealing with the Cahn-Hilliard problem. On the theoretical side, Elliot and Zheng [9] investigate the existence and uniqueness of the solution. On the numerical side, the finite element method [6–8], the finite difference method [14,17,19,25,27], the spectral method [11,16,21,23,30], the finite volume method [10] and the discontinuous Galerkin method [26] have been proposed and applied to solve the Cahn-Hilliard equation numerically. In particular, we mention here the conservation difference scheme with unconditional stability, see, e.g., [14,29].

As for the time discretization aspect, it is noted that the numerical simulation of Cahn-Hillard model needs a long time to reach the steady state. Therefore, the schemes with high stability that allows large time-stepping are very useful to reduce the total computation time. In [17], a semi-implicit difference scheme with an extra term is proposed which allows large time-stepping for long-term simulation. In [15,29], a time adaptivity strategy based on the energy evolution are presented. For more information, please refer to [20].

In this paper, we present a new type of adaptive time-stepping scheme which only requires to input the minimum time step, the maximum time step and the tolerance of the energy decay rate. An energy identity, which connects the energy variation and the space gradient of the solution, is used to predict the time-step. Using the energy identity for the adaptive time-stepping method can avoid the difficulty of choosing artificial parameters needed by some other existing methods. With the predicted time steps, we aim at equi-distributing the energy curve in each time step. By setting the minimal time step and by adjusting the quantity of the energy decay rate, the accuracy of the numerical solution is pre-determined. Moreover, we consider two kinds of difference schemes to demonstrate that they can be integrated into the adaptive time-stepping framework.

The outline of this paper goes as follows. In Section 2, we briefly review the finite difference scheme proposed in [29] as well as some of its important properties. Based on these results, Section 3 is devoted to deriving an adaptive time-stepping algorithm. In Section 4, the time-stepping method is applied to the linear scheme based on the convex splitting and a nonlinear scheme based on the Crank-Nicolson scheme. Some numerical experiments are carried out to show the effectiveness of the algorithm in Section 5. Some concluding remarks will be given in the final section.

2 Model and discretization

The Cahn-Hilliard equation is the gradient flow of the energy functional

$$E(u)=\frac{\kappa}{2}\|\nabla u\|^2+\frac{1}{4}\|u^2-1\|^2, \tag{2.1}$$