COMMUNICATIONS IN COMPUTATIONAL PHYSICS
Vol. 5, No. 2-4, pp. 836-848

Commun. Comput. Phys.
February 2009

Third Order WENO Scheme on Three Dimensional Tetrahedral Meshes

Yong-Tao Zhang[1,*] and Chi-Wang Shu[2]

[1] *Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556-4618, USA.*
[2] *Division of Applied Mathematics, Brown University, Providence, RI 02912, USA.*

Received 29 August 2007; Accepted (in revised version) 25 January 2008

Available online 1 August 2008

Abstract. We extend the weighted essentially non-oscillatory (WENO) schemes on two dimensional triangular meshes developed in [7] to three dimensions, and construct a third order finite volume WENO scheme on three dimensional tetrahedral meshes. We use the Lax-Friedrichs monotone flux as building blocks, third order reconstructions made from combinations of linear polynomials which are constructed on diversified small stencils of a tetrahedral mesh, and non-linear weights using smoothness indicators based on the derivatives of these linear polynomials. Numerical examples are given to demonstrate stability and accuracy of the scheme.

AMS subject classifications: 65M99

Key words: Weighted essentially non-oscillatory (WENO) schemes, finite volume schemes, high-order accuracy, tetrahedral meshes.

1 Introduction

The weighted essentially non-oscillatory (WENO) methodology adopted in this paper, for solving hyperbolic conservation laws with discontinuous solutions, was first developed in [9] for a third order finite volume version in one space dimension and in [8] for third and fifth order finite difference version in multi space dimensions with a general framework for the design of the smoothness indicators and non-linear weights. The main idea of the WENO scheme is to form a weighted combination of several local reconstructions based on different stencils (usually referred to as small stencils) and use it as the final WENO reconstruction. The combination coefficients (also called non-linear weights) depend on the linear weights, often chosen to increase the order of accuracy over that on each small stencil, and on the smoothness indicators which measure the smoothness of the reconstructed function in the relevant small stencils. WENO schemes have

*Corresponding author. *Email addresses:* `yzhang10@nd.edu` (Y.-T. Zhang), `shu@dam.brown.edu` (C.-W. Shu)

the advantage of attaining uniform high order accuracy in smooth regions of the solution while maintaining sharp and essentially monotone shock transitions. It is more difficult to design WENO schemes for unstructured meshes. First of all, the finite difference approach [8] requires mesh smoothness and cannot be used on unstructured meshes while maintaining local conservation, and we must use the more complicated and more costly finite volume approach [9, 12]. There are two types of WENO schemes designed in the literature on unstructured meshes. The first type consists of WENO schemes whose order of accuracy is not higher than that of the reconstruction on each small stencil. That is, for this type of WENO schemes, the non-linear weights are designed purely for the purpose of stability, or to avoid spurious oscillations, and they do not contribute towards the increase of the order of accuracy. Such WENO schemes are easier to construct, since the linear weights can be chosen as arbitrary positive numbers for better stability, for example the centered small stencil can be assigned a larger linear weight than the others. The WENO schemes in [4] for two dimensional triangulations and in [2, 3] for three dimensional triangulations belong to this class. The second type consists of WENO schemes whose order of accuracy is higher than that of the reconstruction on each small stencil. For example, the third order WENO scheme in [7] is based on second order linear polynomial reconstructions on small stencils, and the fourth order WENO scheme in [7] is based on third order quadratic polynomial reconstructions on small stencils. See also [15] for similar WENO schemes for solving Hamilton-Jacobi equations, which belong to the second type as well. This second type of WENO schemes are more difficult to construct, however they have a more compact stencil than the first type WENO schemes of the same accuracy, which is an advantage in some applications, such as when the WENO methodology is used as limiters for the discontinuous Galerkin methods [10, 11]. In this paper, we generalize the second type WENO schemes in [7] to three dimensions, and construct a third order finite volume WENO scheme on three dimensional tetrahedral meshes. We use the Lax-Friedrichs monotone flux as building blocks, third order reconstructions made from combinations of second order linear polynomials which are constructed on diversified small stencils of a tetrahedral mesh, and non-linear weights using smoothness indicators based on the derivatives of these linear polynomials. Numerical examples are given to demonstrate the stability and accuracy of the scheme.

The organization of this paper is as follows. The algorithm is developed in Sections 2 and 3. Section 4 contains numerical examples verifying stability, convergence and accuracy of the algorithm. Concluding remarks are given in Section 5.

2 The finite volume formulation on 3D tetrahedral meshes

In this paper we solve the three-dimensional conservation law

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}+\frac{\partial g(u)}{\partial y}+\frac{\partial h(u)}{\partial z}=0 \tag{2.1}$$

using the finite volume formulation. Computational control volumes are tetrahedrons.