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Mechanics-Based Solution Verification for Porous Media Models

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Abstract. This paper presents a new approach to verify the accuracy of computational simulations. We develop mathematical theorems which can serve as robust *a posteriori* error estimation techniques to identify numerical pollution, check the performance of adaptive meshes, and verify numerical solutions. We demonstrate performance of this methodology on problems from flow thorough porous media. However, one can extend it to other models. We construct mathematical properties such that the solutions to Darcy and Darcy-Brinkman equations satisfy them. The mathematical properties include the total minimum mechanical power, minimum dissipation theorem, reciprocal relation, and maximum principle for the vorticity. All the developed theorems have firm mechanical bases and are independent of numerical methods. So, these can be utilized for solution verification of finite element, finite volume, finite difference, lattice Boltzmann methods and so forth. In particular, we show that, for a given set of boundary conditions, Darcy velocity has the minimum total mechanical power of all the kinematically admissible vector fields. We also show that a similar result holds for Darcy-Brinkman velocity. We then show for a conservative body force, the Darcy and Darcy-Brinkman velocities have the minimum total dissipation among their respective kinematically admissible vector fields. Using numerical examples, we show that the minimum dissipation and total mechanical power theorems can be utilized to identify pollution errors in numerical solutions. The solutions to Darcy and Darcy-Brinkman equations are shown to satisfy a reciprocal relation, which has the potential to identify errors in the numerical implementation of boundary conditions. It is also shown that the vorticity under both steady and transient Darcy-Brinkman equations satisfy maximum principles if the body force is conservative and the permeability is homogeneous and isotropic. A discussion on the nature of vorticity under steady and transient Darcy equations is also presented. Using several numerical examples, we will demonstrate the predictive capabilities of the proposed *a posteriori* techniques in assessing the accuracy of numerical solutions for a general class of problems, which could involve complex domains and general computational grids.

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1 Introduction

1.1 Validation and verification (V&V)

Errors can arise in both physical modeling and numerical simulation. The study of errors due to physical modeling is referred to as validation, and the study of error in a numerical simulation is referred to as verification. As Blottner [13] nicely puts it, validation is to solve "*right governing equations*" and verification is to solve "*governing equation right*". Validation errors arise when a model is used out of its application range. The errors in the verification, on the other hand, can arise from three broad sources including numerical errors, round-off errors (due to the finite precision arithmetic), and programming mistakes [40]. Basically, the verification is to ensure that the code produces a solution with some degree of accuracy, and the numerical solution is consistent. Verification itself is conducted into two modes: verification of code and verification of calculation [44, 46]. Verification of code addresses the question of whether the numerical algorithms have been programmed and implemented correctly in the code. The two currently popular approaches to verify a code are the *method of exact solutions* (MES) and the *method of manufactured solutions* (MMS). More thorough discussions on MES and MMS can be found in [32, 46, 47].

Verification of calculation (which is also referred to as solution verification) estimates the overall magnitude (not just the order) of the numerical errors in a calculation, and the procedure invariably involves *a posteriori* error estimation [49]. The numerical errors in the solution verification can arise from two different sources including discretization errors and solution errors. The discretization errors refer to all the errors caused by conversion of the governing equations (PDEs and boundary conditions) into discrete algebraic equations whereas the solution errors refer to the errors in approximate solution of the discrete equations. The numerical errors may arise from insufficient mesh resolution, improper selection of time-step, and incomplete iterative convergence. For more details on verification of calculation, see [6, 39, 40, 44–47, 49].

1.2 *A posteriori* techniques

The aim of *a posteriori* error estimation is to assess the accuracy of the numerical approximation in the terms of *known* quantities such as geometrical properties of computational grid, the input data, and the numerical solution. *A posteriori* error techniques monitor various forms of the error in the numerical solution such as velocity, stress, mean fluxes, and drag and lift coefficients [11]. Such error estimation differ from *a priori* error estimates in that the error controlling parameters depend on *unknown* quantities. *A priori*