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An Element Decomposition Method for the Helmholtz Equation

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Abstract. It is well-known that the traditional full integral quadrilateral element fails to provide accurate results to the Helmholtz equation with large wave numbers due to the "pollution error" caused by the numerical dispersion. To overcome this deficiency, this paper proposed an element decomposition method (EDM) for analyzing 2D acoustic problems by using quadrilateral element. In the present EDM, the quadrilateral element is first subdivided into four sub-triangles, and the local acoustic gradient in each sub-triangle is obtained using linear interpolation function. The acoustic gradient field of the whole quadrilateral is then formulated through a weighted averaging operation, which means only one integration point is adopted to construct the system matrix. To cure the numerical instability of one-point integration, a variation gradient item is complemented by variance of the local gradients. The discretized system equations are derived using the generalized Galerkin weakform. Numerical examples demonstrate that the EDM can achieves better accuracy and higher computational efficiency. Besides, as no mapping or coordinate transformation is involved, restrictions on the shape elements can be easily removed, which makes the EDM works well even for severely distorted meshes.

AMS subject classifications: 65C20, 65N30, 65N22, 68U20

Key words: Helmholtz equation, numerical methods, element decomposition method, one-point integration, quadrilateral element, dispersion error.

1 Introduction

With the increasing demands on the acoustic performance of enclosed cavities, such as the automotive passenger compartments and aircraft cabins, careful considerations are

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needed in designing these sophisticated products. The vibration noise is essentially a wave propagation phenomenon governed by the Helmholtz equation, and a large amount of research work can be found in the literature reviews [1,2].

As the analytical solution is only available for problems with very simple configurations, numerical methods for solving Helmholtz problems are becoming more and more prevalent. Over the past several decades, many efficient algorithms such as the finite element method (FEM) [3,4], the boundary element method (BEM) [5,6], the wave based method (WBM) [7], the modified smoothed particle hydrodynamics (MSPH) method [8], the hybrid-Trefftz method [9], the smoothed finite element method (SFEM) [10, 11] and the weak Galerkin (WG) method [12], etc., have already been proposed by researchers to simulate the acoustic scattering. Among these different methodologies, the standard FEM is still so far the most well-developed and widely-used numerical tool in predicting the acoustic scattering. Quadrilateral isoparametric element, which is a primary element of FEM, plays a very important role in noise simulation. However, a well-known deficiency associated with the use of quadrilateral element for the Helmholtz equation is the loss of ellipticity as the wave number increases [13], which will inevitably leads to the "pollution error". At higher frequencies, the dispersion error tends to accumulate, resulting in completely erroneous results. Although the use of more elements per wavelength can improve the numerical accuracy in acoustic analysis, the sufficiently refined mesh will leads to a dramatic increase of computational cost, especially for large scale practical acoustic problems. The accuracy of traditional quadrilateral element is also significantly depends on the mesh quality. As pointed in [14], a severely distorted mesh will cause a larger "pollution error". Since isoparametric coordinate transformation is involved in formulating the system equations, violent distorted meshes are not permitted. The third shortcoming inherent in the conventional quadrilateral element is that it suffers from a low computational efficiency. This is because four integration points are adopted when performing the numerical integration, which inevitably increase the computational cost.

Studies have found that the pollution error for one-dimensional problems can be completely eliminated by resorting to analytical solutions [15]. While for higher dimensions, the numerical dispersion cannot be completely avoided [16], but can be reduced. Over the past decades, numerous approaches to alleviating the dispersion error have been proposed by researchers, such as the Galerkin/least-squares (GLS) and Galerkin-gradient/least-square (GGLS) methods [13,17,18], the residual-free bubbles approach (RFB) [19], the multiscale finite element method (MFEM) [20], the discontinuous enrichment method (DEM) [21, 22], the weighted-averaging finite-element method (WA-FEM) [23], the modified integration rules (MIR) method [24], the discontinuous Galerkin method (DGM) [25,26], the partition of unity method (PUM) [27,28], the hybrid-Trefftz quadrilateral finite element model [9] and the finite element-least square point interpolation method (FE-LSPIM) [29, 30]. All these approaches are very promising in controlling the pollution error, and improved the numerical accuracy compared to the conventional quadrilateral element. However, the dispersion error in general two- and three-dimensional acoustic problems still cannot be eliminated properly with these al-