An Energy-Preserving Wavelet Collocation Method for General Multi-Symplectic Formulations of Hamiltonian PDEs

Yuezheng Gong and Yushun Wang[*]

Jiangsu Key Laboratory of NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, P.R. China.

Abstract. In this paper, we develop a novel energy-preserving wavelet collocation method for solving general multi-symplectic formulations of Hamiltonian PDEs. Based on the autocorrelation functions of Daubechies compactly supported scaling functions, the wavelet collocation method is conducted for spatial discretization. The obtained semi-discrete system is shown to be a finite-dimensional Hamiltonian system, which has an energy conservation law. Then, the average vector field method is used for time integration, which leads to an energy-preserving method for multi-symplectic Hamiltonian PDEs. The proposed method is illustrated by the nonlinear Schrödinger equation and the Camassa-Holm equation. Since differentiation matrix obtained by the wavelet collocation method is a cyclic matrix, we can apply Fast Fourier transform to solve equations in numerical calculation. Numerical experiments show the high accuracy, effectiveness and conservation properties of the proposed method.

1 Introduction

A numerical method which can preserve one or more physical/geometric properties of the system exactly is called geometric or structure-preserving integrators. As geometric integration has gained remarkable success in the numerical analysis of ODEs [1, 2], it is believable to extend the idea of geometric integration to significant PDEs. Many conservative PDEs, for instance the sine-Gordon (SG) equation, the nonlinear Schrödinger

(NLS) equation, the Korteweg-de Vries (KdV) equation, the Camassa-Holm (CH) equation, the Maxwell's equations and so on can be rewritten as multi-symplectic Hamiltonian system, which has the properties of multi-symplectic structure, energy and momentum conservation laws [3–5]. To inherit the multi-symplectic structure, many multi-symplectic methods [6–9] are developed in recent years. For more details in applications, please refer to review articles [10,11] and references therein. Except the multi-symplectic conservation law, multi-symplectic Hamiltonian system also has the energy conservation law. The conservation of energy is a crucial property of mechanical systems and plays an important role in the study of properties of solutions [12,13]. It is valuable to expect that energy-preserving discretizations for conservative PDEs will produce richer information on the discrete systems. Li and Vu-Quoc [14] gave a historical survey of energy-preserving methods for PDEs and their applications, especially to nonlinear stability. Energy-preserving methods [15–19] have successfully made many applications. However, these methods have an ad hoc character and are not completely systematic either in their derivation or in their applicability; in contrast, our method discussed here is completely systematic, applied to a huge class of conservation PDEs.

Recently, wavelet-based numerical methods become increasingly popular as they combine the advantages of both spectral method and finite difference method (FDM) [20–22]. Compared with spectral method, wavelet-based methods have good spatial localization and generate a sparse space differentiation matrix, and compared with FDM, wavelet-based methods have good spectral localization and higher order of accuracy. The wavelet-based algorithms can be roughly classified into two categories: wavelet-Galerkin and wavelet collocation. In [23] and [24], Daubechies' compactly supported orthogonal wavelets and second-generation wavelets are proposed to combine with symplectic schemes to construct multiresolution symplectic solvers for wave propagation problems and the method is of wavelet-Galerkin type. However, it is very difficult to deal with nonlinear problems, as it needs the passage between wavelet coefficients and physical space. In order to overcome this difficulty, a wavelet collocation method is proposed in [25], in which the autocorrelation functions of Daubechies compactly supported scaling functions are shown to have the merits of symmetry and nice interpolation properties, and so no extra computation is required for the passage between wavelet coefficients and physical space. Based on the autocorrelation functions of Daubechies compactly supported scaling functions, the symplectic wavelet collocation method and the multi-symplectic wavelet collocation method are developed for Hamiltonian PDEs with periodic boundary conditions in [26] and [27], respectively, in which numerical experiments illustrate the remarkable behavior of the methods.

The average vector field (AVF) method is first written down in [28] and identified as energy-preserving and as a B-series method in [29]. For ordinary differential equation

$$\dot{y} = f(y), \quad y \in \mathbb{R}^d, \tag{1.1}$$