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An Adaptive Grid Method for Singularly Perturbed Time-Dependent Convection-Diffusion Problems

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Abstract. In this paper, we study the numerical solution of singularly perturbed time-dependent convection-diffusion problems. To solve these problems, the backward Euler method is first applied to discretize the time derivative on a uniform mesh, and the classical upwind finite difference scheme is used to approximate the spatial derivative on an arbitrary nonuniform grid. Then, in order to obtain an adaptive grid for all temporal levels, we construct a positive monitor function, which is similar to the arc-length monitor function. Furthermore, the ε-uniform convergence of the fully discrete scheme is derived for the numerical solution. Finally, some numerical results are given to support our theoretical results.

AMS subject classifications: 65L10, 65L12, 65L50

Key words: Uniform convergence, singularly perturbed, convection-diffusion problems, adaptive grid, an upwind finite difference scheme.

1 Introduction

In this paper, we consider the following singularly perturbed time-dependent convection-diffusion problem

$$\begin{cases} u_t(x,t)+\mathcal{L}_{x,\varepsilon}u(x,t)=f(x,t), \quad (x,t)\in G=\Omega\times(0,T]\equiv(0,1)\times(0,T], \\ u(x,0)=u_0(x), \quad x\in\Omega, \\ u(0,t)=u(1,t)=0, \quad t\in(0,T], \end{cases} \tag{1.1}$$

where

$$\mathcal{L}_{x,\varepsilon}u\equiv-\varepsilon u_{xx}+a(x)u_x+b(x)u,$$

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$0<\varepsilon\ll 1$ is a small parameter. It is also assumed that $a(x)$ and $b(x)$ are sufficiently smooth functions, and there exist constants α and β such that

$$a(x)\geq\alpha>0,\qquad b(x)\geq\beta\geq 0,\quad \forall x\in[0,1]. \tag{1.2}$$

Under conditions (1.2) and enough smoothness and compatibility conditions imposed on the functions $u_0(x)$ and $f(x,t)$, the above problem (1.1) has a unique solution which exhibits a boundary layer at $x=1$ (see, e.g. [1]).

Due to the presence of boundary layer, the classical finite difference or finite element methods will not give some satisfactory numerical results on an uniform mesh. Thus, in order to get a reliable numerical solution for (1.1) by using the classical numerical methods, an unacceptably large number of mesh points (ε-dependent) are required. Obviously, the amount of computation in this way is too huge. Therefore, many researchers tried to develop the concept of ε-uniform numerical methods, in which the order of convergence and the error constant are independent of ε.

In the past few decades, many ε-uniform numerical schemes were proposed in the literature for the singularly perturbed problems(see, e.g. [2]). As far as we know, the numerical methods for singularly perturbed problems are widely classified into two categories, namely, the layer-adapted grid approach and the adaptive grid approach. Of these two approaches, ε-uniform numerical results for the first approach is more satisfactory. For this reason, based on the layer-adapted grid approach, there have been some numerical results of the singularly perturbed time-dependent convection-diffusion problems (see, e.g. [3–10]). However, we know that the above layer-adapted grid approach must require a priori information about the location and width of the boundary layer. Obviously, if a priori information is unavailable about the boundary layer, the layer-adapted grid approach will become invalid. Thus, the adaptive grid approach is more popular in solving singularly perturbed problems.

Recently, Gowrisankar and Natesan [11] developed an adaptive grid approach to solve the above problem (1.1). However, on each time level, they must use an adaptive grid algorithm to obtain a non-uniform grid. In other words, they obtained a different spatial mesh on each time level. This drawback motivates to construct a non-uniform grid which is suitable for all time levels, in which the same order of convergence and the error estimation can be established. In [12], in order to obtain a mesh for all time levels, Gowrisankar and Natesan considered equidistribution of $u(x,t)$ at a fixed time T_0, $0<T_0\leq T$, where $u(x,t)$ is the solution of problem (1.1). Then, they used the classical upwind finite difference scheme on this mesh and obtained a first-order rate of convergence for the numerical method. In a word, the monitor functions presented in [11, 12] are much more complex. Therefore, it is always desirable to provide an efficient monitor function and the corresponding mesh generation algorithm in a simplest way.

In this paper, we will study an adaptive grid approach of problem (1.1). Specifically, we first use the backward Euler method on an uniform mesh to discrete the time derivative of problem (1.1). On each time level, we obtain a second-order singularly perturbed ordinary differential equation. Then, we can transform these equations into a system of