Commun. Comput. Phys.
doi: 10.4208/cicp.190615.090516a

Vol. **20**, No. 5, pp. 1359-1380
November 2016

On the Dynamics of the Weak Fréedericksz Transition for Nematic Liquid Crystals

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Received 19 June 2015; Accepted (in revised version) 9 May 2016

Abstract. We propose an implicit finite-difference method to study the time evolution of the director field of a nematic liquid crystal under the influence of an electric field with weak anchoring at the boundary. The scheme allows us to study the dynamics of transitions between different director equilibrium states under varying electric field and anchoring strength. In particular, we are able to simulate the transition to excited states of odd parity, which have previously been observed in experiments, but so far only analyzed in the static case.

AMS subject classifications: 76A15, 74H15, 74H60

Key words: Nematic liquid crystals, Fréedericksz transition, weak anchoring.

1 Introduction

Nematic liquid crystals usually consist of rod-shaped organic molecules for which it is energetically favorable for neighboring molecules to align. This causes macroscopic correlation in the orientation of their long axis, while the molecules themselves are free to flow like a liquid. Nematic liquid crystals have seen widespread use in display devices, due to the optical birefringence associated with the anisotropy of the molecules. Since the orientation of the long axis can be manipulated by applied electromagnetic fields, polarized light can be either stopped by or let through a liquid crystal cell, depending on the applied voltage difference.

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Under the assumption of constant degree of orientation, the state of a nematic liquid crystal is often represented in terms of two linearly independent vector fields: the velocity field giving the flow and the director field giving the local average molecular orientation. In this work we will assume a steady flow field and focus on the dynamics of the director. This implies disregarding phenomena such a back-flow, which can be important in the rheology of liquid crystals [5]. Furthermore, we will restrict our discussion to a one-dimensional liquid crystal cell on $x \in [0,L]$ in the bend-splay geometry. Specifically, we assume the director $\mathbf{n}$ is fixed to the $x-y$ plane, i.e.,

$$\mathbf{n}(x,t) = (\cos(\psi(x,t)), \sin(\psi(x,t)), 0), \tag{1.1}$$

where ψ is the angle between the x-axis and the director. Herein, we will consider numerical solutions to the initial-value problem

$$q\psi_T - \tilde{c}(\psi)(\tilde{c}(\psi)\psi_X)_X + \frac{1}{2}h^2 \sin(2\psi)\tilde{E}^2 = 0, \qquad (X,T) \in (0,1)\times \mathbb{R}^+, \tag{1.2a}$$

$$\psi(X,0) = \psi_0(X), \qquad X \in (0,1), \tag{1.2b}$$

with boundary conditions

$$\psi_X + \frac{1}{2}\frac{\beta}{\tilde{c}^2(\psi)}\sin(2\psi) = 0, \quad X = 0, \tag{1.3a}$$

$$\psi_X - \frac{1}{2}\frac{\beta}{\tilde{c}^2(\psi)}\sin(2\psi) = 0, \quad X = 1. \tag{1.3b}$$

In the above, the dimensionless constants q, h and β represent dissipation, field strength and anchoring strength, respectively, and

$$\tilde{c}(\psi) = \sqrt{\cos^2(\psi) + \frac{\alpha_2}{\alpha_1}\sin^2(\psi)}.$$

The classical example of interaction between the director field and an external electric or magnetic field is the Fréedericksz transition. In its most basic form, it can be described as a competition between elastic torques resisting distortions in the director field, and electromagnetic torques aligning molecules along a preferred direction: Consider e.g. a liquid crystal cell where the easy direction at the surfaces is fixed at $\psi = \pi/2$. The equilibrium configuration is then a homogeneous director field. An electric field is applied twisting the director towards the angle π. When the applied field is below some critical value, $E < E_{\mathrm{F}}$, elastic forces dominate and the constant director state $\psi = \pi/2$ is stable. However, when the field is sufficiently strong, $E > E_{\mathrm{F}}$, the equilibrium state becomes a nontrivial configuration where $\pi/2 < \psi \leq \pi$ in the interior of the domain. This sudden realignment is what is often referred to as the Fréedericksz transition.

Usually, the surfaces of liquid crystals are designed in such a way that the director remains strongly anchored at normal operating voltages in display devices [3]. However,