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A Discrete-Ordinate Discontinuous-Streamline Diffusion Method for the Radiative Transfer Equation

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Abstract. The radiative transfer equation (RTE) arises in many different areas of science and engineering. In this paper, we propose and investigate a discrete-ordinate discontinuous-streamline diffusion (DODSD) method for solving the RTE, which is a combination of the discrete-ordinate technique and the discontinuous-streamline diffusion method. Different from the discrete-ordinate discontinuous Galerkin (DODG) method for the RTE, an artificial diffusion parameter is added to the test functions in the spatial discretization. Stability and error estimates in certain norms are proved. Numerical results show that the proposed method can lead to a more accurate approximation in comparison with the DODG method.

AMS subject classifications: 65N30, 65R20

Key words: Radiative transfer equation, discrete-ordinate method, discontinuous-streamline diffusion method, stability, error estimation.

1 Introduction

The radiative transfer equation, which describes the scattering and absorbing of radiation through a medium, plays an important role in a wide range of applications such as astrophysics, atmosphere and ocean, heat transfer, neutron transport and nuclear physics, and so on. Today, research on the RTE remains to be very active and important, especially in the biomedical optics fields, see e.g. [2, 6, 14, 26, 28].

The RTE can be viewed as a hyperbolic type integro-differential equation. Due to the involvement of both integration and differentiation in the equation, as well as the high dimension of the problem, it is challenging to develop effective numerical methods for

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solving the RTE. The numerical methods can be basically divided into two categories: statistical schemes and deterministic schemes. The interested readers are referred to [11–13, 15, 17, 19, 24, 25, 27].

The discrete-ordinate (DO) method [8, 22, 23], also called the S_N method, is the most popular deterministic method for the RTE, owing to the good compromise among accuracy, flexibility, and moderate computational requirements. This method solves the radiative transfer equation along a discrete set of angular directions, which are the nodal points of a numerical quadrature approximating the integral term on the unit sphere, thus reducing the RTE to a semi-discretized first-order hyperbolic system. To solve the semi-discretized hyperbolic system, it is natural to use the discontinuous Galerkin (DG) discretization, leading to the so-called discrete-ordinate discontinuous Galerkin method. In [16], a DODG method was proposed for the RTE, and error estimates in certain discrete norms were obtained.

The object of this paper is to propose and investigate a discrete-ordinate discontinuous-streamline diffusion method for solving the RTE. Such a method is a combination of the discrete-ordinate technique and the discontinuous-streamline diffusion (DSD) method. The streamline diffusion (SD) finite element method was proposed by Hughes et al. [20] and Johnson et al. [21] in order to cope with the usual instabilities caused by the convection term for the convection–dominated problem. In [3, 4], the streamline diffusion finite element method was analyzed for the multi-dimensional Vlasov-FokkerPlanck system and Fermi pencil beam equation. The DSD method keeps the fundamental structure of the DG method while replacing the Galerkin elements by the SD framework in the upwind iteration procedure. In [9], the DSD method was employed successfully in solving first order hyperbolic problems, where such a modification preserves the advantages of both the upwind approach and the DG method, and also further improves the stability. In this contribution, we seek to improve the DG method for RTE by employing the DSD scheme and derive error estimates of the DODSD method in a norm including the directional gradient. While the DSD approach has been developed and applied to hyperbolic systems or convection–dominated problems, this paper represents the first attempt, to our knowledge, to construct DSD schemes for the RTE. Our numerical results show that the DODSD method can lead to a more accurate solution in comparison with the DODG method.

The rest of this paper is organized as follows. In Section 2, we introduce the RTE and recall a few basic related results. In Section 3, we derive the discrete-ordinate discontinuous-streamline diffusion method, and in Section 4 we present a stability and convergence analysis for the proposed method. Numerical examples are presented in Section 5, illustrating the performance of the numerical method and providing numerical evidence of the theoretical error estimates. Finally, a few concluding remarks are given in Section 6.

Throughout this paper, standard notation is used for Sobolev spaces, and the corresponding semi-norms and norms [10]. Moreover, the letter C denotes a generic positive constant whose value may be different at different occurrences.