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Gradient Recovery for Elliptic Interface Problem: I. Body-Fitted Mesh

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Abstract. In this paper, we propose a new gradient recovery method for elliptic interface problem using body-fitted meshes. Due to the lack of regularity of the solution at the interface, standard gradient recovery methods fail to give superconvergent results and thus will lead to overrefinement when served as *a posteriori* error estimators. This drawback is overcome by designing a new gradient recovery operator. We prove the superconvergence of the new method on both mildly unstructured meshes and adaptive meshes. Several numerical examples are presented to verify the superconvergence and its robustness as *a posteriori* error estimator.

AMS subject classifications: 65L10, 65L60, 65L70

Key words: Elliptic interface problem, gradient recovery, superconvergence, body-fitted mesh, *a posteriori* error estimator, adaptive method.

1 Introduction

Interface problem frequently appears in the fields of fluid dynamics and material science, where the background consists of rather different materials. The numerical challenge comes from discontinuities of the coefficient at the interface, where the solution is not smooth in general. Computational methods for elliptic interface problem have been studied intensively in literature, which can be roughly categorized into two types: unfitted mesh methods and body-fitted mesh methods.

Numerical methods based on unfitted meshes solve interface problems on Cartesian grids, among which, famous examples include immersed boundary method (IBM) by Peskin [38, 39] and immersed interface method (IIM) by Leveque and Li [27], just to name a few. We refer interested readers to [29] for a review of the literature. IBM uses Dirac δ-function to model discontinuity and discretizes it to distribute a singular source to the

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nearest grid point. IIM constructs special finite difference schemes near the interface to get an accurate approximation of the solution. It was further developed in the framework of finite element methods [28, 30, 31], which modifies basis functions on interface elements. Moreover, in [23, 24], a special weak form was derived based on the Petrov-Galerkin method to discretize elliptic interface problem. A shortcoming of unfitted mesh methods is that the resulting discretized linear system is in general non-symmetric and indefinite even though the original continuous problem is self-adjoint.

Body-fitted mesh methods require mesh grids to align with the interface in order to capture discontinuities. The resulting discretized linear system is symmetric and positive definite if the original continuous problem is self-adjoint. Error estimates for finite element method with body-fitted meshes have been established by [2,6,12,45]. In particular, [12] showed that the smooth interface could be approximated by the linear interpolation of distinguished points on the interface. Although the solution to interface problem has low global regularity, the finite element approximation was shown to have nearly the same optimal error estimates in both L^2 and energy norms as for regular (non-interface) problems.

Meanwhile, superconvergence analysis has attracted considerable attention in the community of finite element methods, and theories have been well developed for regular problems [3, 9, 42, 50]. Then it is natural to ask if one can obtain similar superconvergence results for elliptic interface problem. However, limited work has been done in this direction due to the lack of regularity of the solution at the interface. Recently, [13, 14] proposed two special interpolation formulae to recover flux for the one-dimensional linear and quadratic immersed finite element methods. Supercloseness was established between the finite element solution and the linear interpolation of the exact solution in [43].

In this paper, we aim to develop a gradient recovery technique for elliptic interface problem based on the body-fitted finite element discretization. Standard gradient recovery operators, including superconvergent patch recovery (SPR) [51, 52] and polynomial preserving recovery (PPR) [35, 36, 47], produce superconvergent recovered gradient only when the solution is smooth enough. Therefore, they can not be applied directly to elliptic interface problem since the solution has low regularity at the interface due to the discontinuities of the coefficient. Furthermore, building up recovery type *a posteriori* error estimators based on these methods will lead to overrefinement as studied in [8].

An observation that we rely on is that, even though the solution has low global regularity, it is in general piecewise smooth on each subdomain separated by the smooth interface (special case as Kellogg problem will also be discussed in Example 5.4). It motivates us to develop a new gradient recovery method by applying PPR gradient operator on each subdomain since PPR is a local gradient recovery method. One one hand, for a node away from the interface, we use stand PPR gradient recovery operator; On the other hand, for a node close to the interface, we design the gradient recovery operator by fitting a quadratic polynomial in the least-squares sense using the sampling points only in one subdomain. This will generate two approximations of the gradient in each subdomain for a node on the interface, which is consistent with the fact that the solu-