An Improved Formulation for Hybridizable Discontinuous Galerkin Fluid-Structure Interaction Modeling with Reduced Computational Expense

Jason P. Sheldon\textsuperscript{1}, Scott T. Miller\textsuperscript{2} and Jonathan S. Pitt\textsuperscript{1,}* \\
\textsuperscript{1} Applied Research Laboratory, The Pennsylvania State University, University Park, PA 16802, USA. \\
\textsuperscript{2} Sandia National Laboratories, Albuquerque, NM 87185, USA. \\
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Abstract. This work presents two computational efficiency improvements for the hybridizable discontinuous Galerkin (HDG) fluid-structure interaction (FSI) model presented by Sheldon et al. A new formulation for the solid is presented that eliminates the global displacement, resulting in the velocity being the only global solid variable. This necessitates a change to the solid-mesh displacement coupling, which is accounted for by coupling the local solid displacement to the global mesh displacement. Additionally, the mesh basis and test functions are restricted to linear polynomials, rather than being equal-order with the fluid and solid. This change increases the computational efficiency dynamically, with greater benefit the higher order the computation, when compared to an equal-order formulation. These two improvements result in a 50\% reduction in the number of global degrees of freedom for high-order simulations for both the fluid and solid domains, as well as an approximately 50\% reduction in the number of local fluid domain degrees of freedom for high-order simulations. The new, more efficient formulation is compared against that from Sheldon et al. and negligible change of accuracy is found.

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1 Introduction

Many natural phenomena, such as blood transport or aeroelastic flutter [1], require tightly coupled fluid-structure interaction (FSI) simulations to accurately represent the complex multi-physics. FSI simulations, particularly those involving large amplitude, low
frequency, solid deformation, are substantially more computationally expensive and dif-
ficult to model than traditional fluid-only or solid-only simulations due to the tight cou-
pling on fluid-solid interface and the disparate mathematics used to describe the fluid
and solid regions in the models [2]. With the goal of reducing the computational expense
associated with FSI simulations, the hybridizable discontinuous Galerkin (HDG) method
was first utilized for FSI simulation by Sheldon et al. [3, 4]. The HDG method is a re-
cently developed finite element method that has the benefits of discontinuous Galerkin
(DG) methods, such as scalability in parallel, local conservation of variables, stability
with complicated geometries, and high-order accuracy based on the degree of the ap-
proximating polynomial [5], while reducing the high computational cost associated with
DG methods [6]. This reduction is achieved by separation (hybridization) of the solution
into local element solutions and global trace solutions on the element interfaces. This
minimizes the number of globally coupled degrees of freedom (DOFs), along with the
computational cost of solving the global system, while preserving the benefits of DG
methods. Only the primary variable of interest, referred to as the hybrid unknown, exists
in both the local and global solution spaces, and it is the only unknown solved for glob-
ally, i.e., across the entire mesh, in this formulation. The individual local variables can be
solved in an inherently parallel fashion, being completely decoupled from one another.

This work does not focus on deriving formulations with the HDG method or on FSI in
general. For more on these topics, please see Sheldon et al. [3, 4, 7]. Additionally, Nguyen
et al. present an excellent introduction to the HDG method in [8] for Stokes flow and in [9]
for an overview to a wide variety of physics. Extensive literature exists on a multitude
of other computational methods for FSI, e.g. [10–12], and the interested reader is referred
to the references therein for further information. This work also does not tackle many of
the items identified as future work for HDG FSI in [3, 4, 7], including: higher order post-
processing, the non-linear solid strain’s suboptimal convergence rate, parallel processing
scaling and efficiency studies, investigation of optimal stabilization parameters for HDG
multiphysics, and a full three-dimensional HDG FSI study. These remain interesting and
important areas of future work, each worthy of their own investigation.

Instead, the focus of this work is a reduction in the number of DOFs for the HDG
FSI system and the resultant increase in computational efficiency over the formulations
presented in [3, 4]. Two factors contribute to this DOF reduction. First, the global solid
displacement is eliminating from the solid formulation. This has repercussions on the
solid-mesh FSI coupling that are discussed later. Second, the function spaces for the mesh
test and basis functions are restricted to linear polynomials, regardless of the order of the
rest of the simulation. The computational effects of these two changes is a 50% reduction
in the number of global DOFs for high-order simulations across both the fluid and solid
domains, as well as an approximately 50% reduction in the number of local fluid domain
DOFs for high-order simulations. The specific DOF reductions versus simulation order
are detailed in Appendix A and Table 4.

In the following section a brief background of the mathematics necessary for HDG
FSI modeling is presented. Next, the fluid, solid, and mesh formulations from [3, 4] are
reproduced, but not derived (see [3, 4] for full derivation), and then the modifications to these formulations discussed above are presented. With the new formulations, the same code-to-code comparisons performed in [3, 4] are repeated and the results are shown to be indistinguishable, except for a substantial DOF reduction. Finally, a brief discussion of stability is presented along with some concluding remarks.

2 Background mathematics

A mathematical model of FSI requires governing equations for both fluid systems and solid systems, along with the coupling that combines them. The fluid formulation presented herein is the unsteady incompressible Navier-Stokes equations, cast into the Arbitrary Lagrangian Eulerian (ALE) framework. For the solid equations, a Saint Venant-Kirchhoff non-linear elastodynamics formulation is presented, while a linear elastostatics formulation is used for mesh motion. This last formulation, with arbitrary (user chosen) material properties, is used to update the computational domain on which the fluid is solved (necessary for the ALE Navier-Stokes formulation). Before these formulations themselves can be presented, a certain level of background information is necessary.

2.1 Kinematics

The modeling process for fluids and solids is very similar; however, there are some differences that need to be addressed. Fluid equations are typically written in the Eulerian (deformed) reference frame, while solid equations are generally written in the Lagrangian (referential) reference frame. There is a disconnect between these frames on the fluid-solid interface that can be accounted for in various ways, such as the immersed boundary method [13, 14] or the ALE method [15, 16]. In this work, the ALE formulation is used for the fluid, which accounts for the change in frame by transforming the standard Eulerian governing differential equations into a form defined on the reference configuration \( \Omega_F(0) \). There are many references that provide more details on kinematics and reference frames, such as Gurtin [17] or Spencer [18]; however, for this work it is worth specifically noting the following.

The position of a material point in some reference body \( \beta \) (before deformation) is \( X \), while the position of a material point in a deformed configuration \( \beta \), known as the spatial point, is \( x \). The deformation function \( \chi(X,t) \) maps a material point \( X \) to the spatial point \( x \) at the instant of time \( t \). Considering a smooth sequence of configurations ordered in time, a motion is defined as

\[
\chi(X,t) = x(X,t) = X + u, \tag{2.1}
\]

where \( u \) is the displacement vector, given by

\[
u = x(X,t) - X. \tag{2.2}\]
Operators with capitalized first letters (e.g. \( \text{Grad} \)) denote differential operators with respect to the referential/material coordinate \( X \), while those with lower-case first letters (e.g. \( \text{grad} \)) denote differential operators with respect to the spatial/deformed coordinate \( x \). The deformation-gradient can be expressed as

\[
F(X,t) = \text{Grad} \chi = I + \text{Grad} u, \tag{2.3}
\]

and the determinant of \( F \) is

\[
J = \det(F). \tag{2.4}
\]

In Section 3.1, \( F \) and \( J \) are used for push-forward and pull-back operations in the ALE description of the fluid region.

### 2.2 FSI

For FSI, the displacement vector is piecewise defined over the solid and fluid subregions as

\[
\mathbf{u} = \begin{cases} 
\mathbf{u}_s, & \forall \mathbf{X} \in \Omega_S, \\
\mathbf{u}_m, & \forall \mathbf{X} \in \Omega_F,
\end{cases} \tag{2.5}
\]

where \( \Omega_S \) is the solid subdomain, \( \Omega_F \) is the fluid subdomain, \( \mathbf{u}_s \) is the solid displacement and \( \mathbf{u}_m \) is the (arbitrary) displacement of the (mesh) fluid subdomain. The displacement of the fluid subdomain is governed by the mesh motion, hence the subscript “m” and not “f”, where the subscripts “m”, “s”, and “f” refer to “mesh”, “solid”, and “fluid” respectively. Continuity of displacement on the fluid-solid interface requires

\[
\mathbf{u}_s - \mathbf{u}_m = 0, \quad \forall \mathbf{X} \in \Gamma_{FS}, \tag{2.6}
\]

where \( \Gamma_{FS} = \Omega_S \cap \Omega_F \) is the interface between the solid and fluid subdomains.

### 2.3 Spatial discretization

Regardless of the differential equations being solved, the process of spatial discretization is always roughly the same. Given an arbitrary spatial domain \( \Omega \) over which some governing equation should be solved, break that domain into a sufficiently regular collection \( \mathcal{T}^h \) of disjoint elements \( K \), with faces \( F \). This collection defines the mesh, or triangulation, which serves as a model for the original domain. The governing equations, written in their strong form, have their primary variables approximated by discretized counterparts, are weighted with some arbitrary function, and are integrated over their respective domains, resulting in the weak form of the governing equations. Typically, the test functions (indicated with a tilde) and basis functions for the approximated primary variables (indicated with a superscript \( ^h \)) are chosen from the same finite dimensional solution spaces. An example space for some discretized variable \( a^h \) is

\[
A^h := \left\{ a^h \in \left[ L^2(\mathcal{T}^h) \right]^d : a^h \big|_K \in [P_k(K)]^d, \quad \forall K \in \mathcal{T}^h \right\}, \tag{2.7}
\]
where $d$ is the dimension of a given space, $P_k(K)$ denotes the space of polynomials of at most degree $k$ on element $K$, and $L^2(K)$ denotes the space of square integrable functions on $K$.

### 2.4 Temporal discretization

Temporal discretization is not the focus of this work. See [19] (or many other sources) for derivation, discussion, and presentation of a variety of temporal discretization methods. All results presented in this work use the second-order backwards difference formula (BFD2), which approximates the solution to

$$\frac{dy(t)}{dt} = f(y(t), t), \quad (2.8)$$

as

$$3y_{n+2} - 4y_{n+1} + y_n = 2\Delta tf(t_{n+2}), \quad (2.9)$$

where the subscripts indicate information from various timestep levels.

### 3 Fluid-structure interaction model

In this section the model for HDG FSI is presented. First, the fluid governing equations are presented in their ALE form, while the mesh and solid governing equation are presented in their Lagrangian forms. Second, the constraints necessary for FSI coupling are discussed and the function spaces for the discretized variables are presented. Finally, the HDG formulations from [3,4] are reproduced along with their DOF reduced counterparts. The derivations for the following formulations are only presented when derivations differ from those presented in [3,4]. Commentary is made throughout the entire section on the changes necessary for the DOF reduction. The formulations in this section loosely follow the work of Nguyen et al. [9,20], Soon et al. [21], and Kronbichler et al. [22], with specific attributions given in the full derivations found in [3,4].

#### 3.1 Governing equations

##### 3.1.1 ALE formulation of Navier-Stokes equations

Presented below are the strong forms of the ALE incompressible Navier-Stokes equations, including the conservation of linear momentum, a kinematic compatibility equation for the velocity-gradient, the continuity equation, and appropriate boundary conditions. The primary fluid variables being solved for are the velocity $v$ (hybrid), the
pressure \( p \), and the velocity gradient \( L \).

\[
\rho_i \frac{\partial v_t}{\partial t} + \rho_i L_i \left[ v_t - v_m \right] + \text{Grad} \left( p_i I - \mu_i L_i \right) : F^{-T}_m = f_t \quad \forall X \in \Omega_F(0), \quad (3.1)
\]

\[
L_i - \text{Grad} v_t F^{-1}_m = 0 \quad \forall X \in \Omega_F(0), \quad (3.2)
\]

\[
F^{-T}_m : \text{Grad} v_t = 0 \quad \forall X \in \Omega_F(0), \quad (3.3)
\]

\[
v_t = g_{D_t} \quad \forall X \in \Gamma_{D_t}(0), \quad (3.4)
\]

\[
(p_i I - \mu_i L_i) F^{-T}_m \left| n_i \right| = g_{N_i} \quad \forall X \in \Gamma_{N_F}(0). \quad (3.5)
\]

3.1.2 Linear elastostatics formulation (mesh motion)

For mesh motion, a linear elastostatics formulation is used, with arbitrary (user defined) material properties. The primary variables of displacement \( u \) (hybrid) and deformation gradient \( F \) are chosen for their convenience in the ALE fluid formulation in FSI. With these primary variables, linear elastostatics is governed by the steady form of the conservation of linear momentum, written in the Lagrangian reference frame, along with a constitutive relation for deformation-gradient and appropriate boundary conditions:

\[
- \text{Div} C_m \left[ F_m - I \right] = f_m \quad \forall X \in \Omega_F(0), \quad (3.6)
\]

\[
F_m - I - \text{Grad} u_m = 0 \quad \forall X \in \Omega_F(0), \quad (3.7)
\]

\[
u_m = g_{D_m} \quad \forall X \in \Gamma_{D_F}(0), \quad (3.8)
\]

\[
(C_m \left[ F_m - I \right]) \left| n_m \right| = g_{N_m} \quad \forall X \in \Gamma_{N_F}(0), \quad (3.9)
\]

where \( C_m \) is the elasticity tensor with arbitrary mesh material properties.

3.1.3 Nonlinear elastodynamics formulation

The (geometrically) nonlinear Saint Venant-Kirchhoff model was used in [3, 4] because it was the model Turek and Hron used for their FSI benchmark [23], with which these formulations are compared in Section 4. In [3, 4] the primary variables were the displacement \( u \) (hybrid), the velocity \( v \) (hybrid) and the Green-St. Venant strain \( E \); however, in this work, only the velocity is a hybrid variable, which requires changes to the governing equations. First, the strong form from [3, 4] is presented, which is analogous to the system for linear elastostatics, but with an additional kinematic compatibility condition:

\[
\rho \frac{\partial v_s}{\partial t} - \text{Div} \left[ F_s C_s(E_s) \right] = f_s \quad \forall X \in \Omega_S, \quad (3.10)
\]

\[
E_s - \frac{1}{2} \left( \text{Grad} u_s + (\text{Grad} u_s)^T + (\text{Grad} u_s)^T \text{Grad} u_s \right) = 0 \quad \forall X \in \Omega_S, \quad (3.11)
\]

\[
\frac{\partial u_s}{\partial t} = v_s \quad \forall X \in \Omega_S, \quad (3.12)
\]

\[
u_s = g_{D_s} \quad \forall X \in \Gamma_{D_S}, \quad (3.13)
\]

\[
\left[ F_s C_s(E_s) \right] \left| n_s \right| = g_{N_s} \quad \forall X \in \Gamma_{N_S}, \quad (3.14)
\]
where $C_s$ is the elasticity tensor for the solid’s material properties. The solid deformation tensor $F_s = I + \text{Grad} u_s$ is not a primary variable, but it is shown in the formulation for brevity.

Having a two-hybrid-field formulation is not desirable for several reasons, with the most important reason being that it increases the number of global DOFs, which is contrary to the entire point of the HDG method. At the time [3, 4] were written, no alternative to a two-hybrid-field solid formulation was thought possible; however that has since changed. In order to write a one-hybrid-field solid formulation, Eq. (3.11) must be differentiated with respect to time and Eq. (3.13) must be written in terms of velocity, resulting in

$$
\frac{\partial E_s}{\partial t} - \text{Sym}(\text{Grad} v_s) + \frac{1}{2}[(\text{Grad} v_s)^T \text{Grad} u_s + (\text{Grad} u_s)^T \text{Grad} v_s] = 0 \quad \forall X \in \Omega_s, \quad (3.15)
$$

$$
v = g_{D_s} \quad \forall X \in \Gamma_{D_s}. \quad (3.16)
$$

No other changes to the strong form of the equations are necessary, but more changes will be presented for both the FSI coupling and the weak forms of the solid and mesh governing equations in the following sections.

### 3.2 FSI Coupling

To couple these three formulations together for monolithic FSI, the system requires three conditions on the fluid-solid interface: the solid displacement governs the mesh displacement, the solid and fluid velocity fields are continuous, and finally, the solid and fluid tractions are continuous. For the HDG method, boundary (or interface) conditions need to be applied to the solution trace (the global variables). This was the reason for the two-hybrid-field solid formulation presented in [3, 4]. Those boundary conditions are reproduced below, and then the modification for a one-hybrid-field solid formulation is discussed:

$$
(\mu_m - \mu_s) \big|_{\Gamma_{FS}} = 0, \quad (3.17)
$$

$$
(v_s - v_f) \big|_{\Gamma_{FS}} = 0, \quad (3.18)
$$

$$
(\hat{T}_s [n_s] + \hat{T}_f [n_f]) \big|_{\Gamma_{FS}} = 0, \quad (3.19)
$$

where $\mu$, $v$, and $\hat{T}$ are the global components of the displacement, velocity and traction respectively. The orientation of the fluid-solid interface $\Gamma_{FS}$ requires $n_s = -n_f$. Section 3.4 discusses how the global traction is approximated in terms of the other variables to reduce the number of global unknowns.

Unlike the velocity and traction conditions, which are two-way coupled, the solid-mesh displacement condition (3.17) must be one-way coupled. The mesh should not directly influence the solid because the mesh is non-physical and its properties are arbitrary,
otherwise the artificial mesh ‘stiffness’ would limit the motion of the solid. Therefore, the solid-mesh displacement condition (3.17) must be added to the mesh governing equations on the interface, but not to the solid governing equations. Due to this restriction, it is possible to rewrite Eq. (3.17) in terms of the global mesh displacement, but only the local solid displacement:

\[
\left( \mu_m - u_s \right) \bigg|_{\Gamma_{fs}} = 0,
\]

which is the entire reason a one-hybrid-field solid formulation is possible.

### 3.3 Function spaces

The discontinuous finite element approximation spaces for the discrete fluid, solid, and mesh subproblems presented in [3, 4] were chosen from the spaces reproduced below. The test functions are chosen from the same respective spaces as the discretized primary variables, except for those with Dirichlet boundary conditions. These spaces, as with all the basis functions in this document, are indicated with a tilde.

\[
\begin{align*}
U_{m,s}^h &:= \left\{ u_{m,s}^h \in L^2(\mathcal{T}_{fs}^h) : u_{m,s}^h \big|_K \in [P_k(K)]^d, \ \forall K \in \mathcal{T}_{fs}^h \right\}, \\
V_{fs}^h &:= \left\{ v_{fs}^h \in L^2(\mathcal{T}_{fs}^h) : v_{fs}^h \big|_K \in [P_k(K)]^d, \ \forall K \in \mathcal{T}_{fs}^h \right\}, \\
\mathcal{E}_{s}^h &:= \left\{ E_s^h \in L^2(\mathcal{T}_{s}^h) : E_s^h \big|_K \in [P_k(K)]^{d \times d}, \ \forall K \in \mathcal{T}_{s}^h \right\}, \\
\mathcal{F}_{m}^h &:= \left\{ F_m^h \in L^2(\mathcal{T}_{f}^h) : F_m^h \big|_K \in [P_k(K)]^{d \times d}, \ \forall K \in \mathcal{T}_{f}^h \right\}, \\
\mathcal{L}_{t}^h &:= \left\{ L_t^h \in L^2(\mathcal{T}_{t}^h) : L_t^h \big|_K \in [P_k(K)]^{d}, \ \forall K \in \mathcal{T}_{t}^h \right\}, \\
P_{t}^h &:= \left\{ p_t^h \in L^2(\mathcal{T}_{t}^h) : p_t^h \big|_K \in P_k(K), \ \forall K \in \mathcal{T}_{t}^h \right\}, \\
M_{m,s}^h &:= \left\{ m_{m,s}^h \in L^2(\mathcal{F}_{fs}^h) : m_{m,s}^h \big|_F \in [P_k(F)]^d, \ \forall F \in \mathcal{F}_{fs}^h, m_{m,s}^h \big|_{\partial \Omega_{fs}} = g_{D_{fs}} \right\}, \\
\bar{M}_{m,s}^h &:= \left\{ \bar{m}_{m,s} \in L^2(\mathcal{F}_{fs}^h) : \bar{m}_{m,s} \big|_F \in [P_k(F)]^d, \ \forall F \in \mathcal{F}_{fs}^h, \bar{m}_{m,s} \big|_{\partial \Omega_{fs}} = 0 \right\}, \\
V_{fs}^h &:= \left\{ v_{fs}^h \in L^2(\mathcal{F}_{fs}^h) : v_{fs}^h \big|_F \in [P_k(F)]^d, \ \forall F \in \mathcal{F}_{fs}^h, v_{fs}^h \big|_{\partial \Omega_{fs}} = g_{D_{fs}} \right\}, \\
\bar{V}_{fs}^h &:= \left\{ \bar{v}_{fs} \in L^2(\mathcal{F}_{fs}^h) : \bar{v}_{fs} \big|_F \in [P_k(F)]^d, \ \forall F \in \mathcal{F}_{fs}^h, \bar{v}_{fs} \big|_{\partial \Omega_{fs}} = 0 \right\}, \\
\Psi_{t}^h &:= \left\{ \psi_t^h \in L^2(\mathcal{T}_{t}^h) : \psi_t^h \big|_K \in P_0(K), \ \forall K \in \mathcal{T}_{t}^h \right\},
\end{align*}
\]

where $\mathcal{F}^h$ is the set of all faces, for all elements in $\mathcal{T}^h$. In order to reduce the number of total DOFs (both local and global), the basis and test functions for the mesh variables
are restricted to only linear polynomials. Additionally, because of the changes made in Eq. (3.16), the space for the global velocity must now include a Dirichlet condition for the solid. These new spaces are presented below, and they replace the respective spaces from Eq. (3.21) for the new reduced DOF formulation.

\[ U_m^1 := \left\{ u_h^m \in \left[ L^2(T^h_f) \right]^d : u_h^m|_K \in [P_1(K)]^d, \quad \forall K \in T^h_f \right\}, \]

\[ F_m^1 := \left\{ F_h^m \in \left[ L^2(T^h_f) \right]^{d \times d} : F_h^m|_K \in [P_1(K)]^{d \times d}, \quad \forall K \in T^h_f \right\}, \]

\[ M_m^1 := \left\{ \mu_h^m \in \left[ L^2(F^h_f) \right]^d : \mu_h^m|_F \in [P_1(F)]^d, \quad \forall F \in F^h_f, \mu_h^m|_{\partial \Omega_{D}} = g_D^m \right\}, \quad (3.22) \]

\[ \tilde{M}_m^1 := \left\{ \tilde{\mu}_m \in \left[ L^2(F^h_f) \right]^d : \tilde{\mu}_m|_F \in [P_1(F)]^d, \quad \forall F \in F^h_f, \tilde{\mu}_m|_{\partial \Omega_{D}} = 0 \right\}, \]

\[ V_{h,s}^1 := \left\{ \upsilon_h^m \in \left[ L^2(F^h_f,s) \right]^d : \upsilon_h^m|_F \in [P_k(F)]^d, \quad \forall F \in F^h_f,s, \upsilon_h^m|_{\partial \Omega_{D,s}} = g_{D,s} \right\}. \]

The restrictions from the mesh function space result in a variable local and global fluid domain DOF reduction up to approximately 50% for high-order problems. This is demonstrated in Appendix A. The accuracy of the solution is not affected by restricting the mesh function spaces to linear polynomials because the mesh is a non-physical construct used solely for the ALE transformation. The mesh only has two requirements: that it satisfies all the boundary conditions† and that is sufficiently smooth. Linear polynomials meet both of these requirements, and so using higher-order polynomials is wasteful in terms of computational resources. In fact, as is discussed later, the only other side effect of the restricted mesh function spaces appears to be increased stability. The numerical studies for this work using linear function spaces for the mesh variables had a more stable solution than those with equal-order mesh function spaces.

### 3.4 HDG discretization for FSI sub-problems

The HDG discretization procedure for the three sets of governing equations from Section 3.1 is presented in full in [3, 4]. In this section, the final HDG formulations are reproduced along with a discussion of the differences pertaining to the one-hybrid-field solid formulation.

#### 3.4.1 Fluid

The fluid governing equations are entirely unchanged from [3, 4]. The discretized primary variables for the ALE Navier-Stokes formulation are the velocity-gradient \( \mathbf{L}^h_f \), the velocity \( \mathbf{v}^h_f \), the pressure \( p^h_f \), the trace of the velocity over the element faces \( t^h_f \), and the

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†For a complex fluid-solid interface, it may be desirable to use equal-order mesh function spaces specifically on the interface to preserve displacement continuity between the solid and mesh. This should not substantially increase the number overall DOFs, but is worth noting.
mean pressure over each element $\psi^h_i$. Each has an associated test function: $\tilde{L}_i, \tilde{v}_i, \tilde{p}_i, \tilde{\psi}_i$, and $\tilde{\psi}_i$. Discretizing Eqs. (3.1), (3.2) and (3.3), weighting by appropriate test functions, and then integrating by parts over a cell yields the local weak form of the ALE Navier-Stokes equations defined over a cell. The full derivation is omitted here (See [3, 4]), but the resultant HDG ALE Navier-Stokes fluid problem is:

**Problem 1** (Fluid sub-problem).

Find $\{L_i^h, v_i^h, p_i^h, \psi_i^h\} \in L_i^h \times V_i^h \times P_i^h \times \psi_i^h$ such that

$$
\left( \tilde{L}_i, \rho \frac{\partial v_i}{\partial t} \right)_{K_i} + \left( \tilde{v}_i, \rho \frac{\partial L_i}{\partial t} \right)_{K_i} + \left( \tilde{v}_i, \frac{\partial v_i}{\partial t} \right)_{K_i} + \left( \nabla \tilde{v}_i, \frac{\partial v_i}{\partial t} \right)_{K_i} = 0,
$$

$$
\left( \tilde{L}_i, \rho \frac{\partial L_i}{\partial t} \right)_{K_i} + \left( \tilde{v}_i, \frac{\partial L_i}{\partial t} \right)_{K_i} = 0,
$$

$$
\left( \tilde{v}_i, \rho \frac{\partial L_i}{\partial t} \right)_{\Gamma_{FSI}} + \left( \tilde{v}_i, \frac{\partial L_i}{\partial t} \right)_{\Gamma_{FSI}} = 0,
$$

$$
\left( \tilde{v}_i, \rho \frac{\partial L_i}{\partial t} \right)_{\partial \Omega_i} = 0,
$$

$$
\left( \tilde{v}_i, \rho \frac{\partial L_i}{\partial t} \right)_{\partial \Omega_i} = 0,
$$

and

$$
p_i^0 + p_i^k = p_i^h,
$$

$$
\tilde{T}[n_i]^h := -\mu L_i \frac{\partial v_i}{\partial t} \left| n_i \right|^h + S_f \left( v_i^h - v_i^h \right),
$$

$$
\tilde{T}[n_i]^h := L_i \left[ -\mu L_i^h + L_i^h \left( p_i^0 + \psi_i^h \right) \right] \frac{\partial v_i}{\partial t} \left| n_i \right|^h + S_f \left( v_i^h - v_i^h \right),
$$

$$
S_f := \left( \frac{\mu L_i}{\rho} + p_i \right)_{12}.
$$

It is worth noting that the pressure was decomposed by

$$
p_i^h := p_i^0 + p_i^k, \quad \text{s.t.} \quad \text{Grad } p_i^0 = 0, \quad \text{and} \quad \int_V p_i^k dV = 0, \quad (3.23)
$$

where $p_i^0$ and $p_i^k$ are from the following spaces:

$$
Q^0 := \left\{ p_i^0 \in L^2 \left( \mathcal{T}^h \right) : p_i^0|_K \in P_0(K), \forall K \in \mathcal{T}^h \right\},
$$

$$
Q^k := Q^h \setminus Q^0.
$$

(3.24)
Also, to minimize the number of global DOFs, approximations of the velocity-gradient
trace and pressure trace from the traction \( \hat{T} \) were defined in terms of the other variables
and a stabilization parameter \( S \) and a characteristic length scale \( l \). This is also done for
the solid and mesh formulations.

### 3.4.2 Solid

The solid governing equations are where the major differences in formulation occur. Be-
low, the two-hybrid-field formulation from \[3, 4\] is reproduced, followed by the new one-
hybrid-field formulation. The discretized primary variables for solid formulation are the
Green-St. Venant strain \( E^s \), the displacement \( u^h_s \), the velocity \( \dot{v}^h_s \), the trace of the displacement
over the element faces \( \mu^h_s \) (only in the old two-hybrid-field formulation), and the
trace of the velocity over the element faces \( \dot{v}^h_s \). Each has an associated test function: \( \tilde{E}_s \),
\( \tilde{u}_s \), \( \tilde{v}_s = \tilde{v}^h_s \), and \( \tilde{\mu}_s \). For the two-hybrid-field formulation, an added constraint is necessary
that specifies that the global displacement equals the local displacement on the element
faces. This constraint is not necessary in the new one-hybrid-field formulation.

**Problem 2** (Two-hybrid-field solid sub-problem).

Find \( \{ u^h_s, v^h_s, E^h_s, \mu^h_s, v^h_s \} \in U^h_s \times V^h_s \times E^h_s \times M^h_s \times \mathbf{V}^h_s \), such that

\[
\begin{align*}
  \left( \tilde{v}_s, \partial \frac{\partial v^h_s}{\partial t} \right)_{K_s} + \left( \text{Grad}\tilde{v}_s, F_s C_s \left( E^h_s \right) \right)_{K_s} + \left( \tilde{v}_s, \hat{T}[n]^h[s] \right)_{\partial K_s} &= \left( \tilde{v}_s, f_s \right)_{K_s}, \\
  \left( \tilde{E}_s, E^h_s \right)_{K_s} - \left( \text{Sym}(E) \right)_{K_s} \text{Grad} u^h_s - \left( \frac{1}{2} \tilde{E}_s, \text{Grad} u^h_s \right) \text{Grad} u^h_s \right)_{K_s} &+ \left( \text{Sym}(E) \right)_{K_s} \left( \mu^h_s - \mu^h_s \right) \right)_{\partial K_s} = 0, \\
  \left( \tilde{u}_s, \frac{\partial u^h_s}{\partial t} \right)_{K_s} - \left( \tilde{u}_s, v^h_s \right)_{K_s} &= 0, \\
  \left( \tilde{\mu}_s, u^h_s \right)_{\partial K_s} - \left( \tilde{\mu}_s, \mu^h_s \right)_{\partial K_s} &= 0, \\
  \left( \tilde{v}_s, \hat{T}[n]_{\partial T^h[s]^{\Gamma_{FSI}}} + \left[ \tilde{v}_s, \hat{T}[n]_{\Gamma_{FSI}} + \hat{T}[n]_{I} \right]_{\Gamma_{FSI}} + \left( \tilde{v}_s, S_s \left( v^h_s - v^h_s \right) \right)_{\Gamma_{FSI}} \right) = \left( \tilde{v}_s, g_{N_s} \right)_{\Gamma_{N_s}}, \\
  \forall \{ \tilde{u}_s, \tilde{v}_s, \tilde{E}_s, \tilde{\mu}_s, \tilde{\nu}_s \} \in U^h_s \times V^h_s \times E^h_s \times M^h_s \times \mathbf{V}^h_s,
\end{align*}
\]

where

\[
\begin{align*}
  F_s &= I + \text{Grad} u^h_s, \\
  \hat{T}[n]_{s} &= - \left[ F_s C_s \left( \hat{E}^h_s \right) \right] n_s + S_s \left( v^h_s - v^h_s \right), \\
  S_s &= \frac{\epsilon_s}{l_s} I.
\end{align*}
\]
Problem 3 (One-hybrid-field solid sub-problem).
Find \( \{ u^h_s, v^h_s, E^h_s, v^h_s \} \in U^h_s \times V^h_s \times E^h_s \times V^h_s \), such that

\[
\begin{align*}
\left( \tilde{v}_s, \frac{\partial v^h_s}{\partial t} \right)_{K_s} & + \left( \text{Grad} \tilde{v}_s, F_s C_s \left( E^h_s \right) \right)_{K_s} + \left( \tilde{v}_s, \hat{T}[n]_s \right)_{\partial K_s} = (\tilde{v}_s, f_s)_{K_s}, \\
\left( \tilde{E}_s, \frac{\partial E^h_s}{\partial t} \right)_{K_s} & - \left( \text{Sym}(E)_s, \text{Grad} v^h_s \right)_{K_s} - \left( \frac{1}{2} \tilde{E}_s, (\text{Grad} v^h_s)^T \text{Grad} u^h_s \right)_{K_s} \\
& - \left( \frac{1}{2} \tilde{E}_s, (\text{Grad} u^h_s)^T \text{Grad} v^h_s \right)_{K_s} + \left( \text{Sym}(E)_s, n, (v^h_s - v^h_s) \right)_{\partial K_s} = 0, \\
\left( \tilde{u}_s, \frac{\partial u^h_s}{\partial t} \right)_{K_s} & - (\tilde{u}_s, v^h_s)_{K_s} = 0, \\
\left( \tilde{v}_s, \hat{T}[n]_s \right)_{\partial T^h_s \setminus \Gamma_{FSI}} + \left( \tilde{v}_s, \hat{T}[n]_s + \hat{T}[n]_t \right)_{\Gamma_{FSI}} + \left( \tilde{v}_s, S_s (v^h_s - v^h_s) \right)_{\Gamma_{FSI}} = (\tilde{v}_s, g_{N_s})_{\Gamma_{N_s}}, \\
\forall \{ \tilde{u}_s, \tilde{v}_s, \tilde{E}_s, \tilde{v}_s \} \in U^h_s \times V^h_s \times E^h_s \times V^h_s,
\end{align*}
\]

where

\[
F_s = I + \text{Grad} u^h_s, \\
\hat{T}[n]_s := - \left[ F_s C_s \left( E^h_s \right) \right] n_s + S_s (v^h_s - v^h_s), \\
S_s := \frac{H_s}{I_s} I.
\]

3.4.3 Mesh

Finally, the mesh governing equations are presented in almost exactly the same form as they were in [3,4], with the exception of the change to the solid-mesh displacement condition from Eq. (3.20) and the restricted function spaces from Eq. (3.22). The discretized primary variables for mesh formulation are the deformation-gradient \( F^h_m \), the displacement \( u^h_m \), and the trace of the displacement over the element faces \( \mu^h_m \). Each has an associated test function: \( \hat{F}_m, \hat{u}_m, \) and \( \hat{\mu}_m \).

Problem 4 (Mesh sub-problem).
Find \( \{ u^h_m, F^h_m, \mu^h_m \} \in U^1_m \times F^1_m \times M^1_m \) such that

\[
\begin{align*}
\left( \text{Grad} \hat{u}_m, C_m \left( F^h_m \right) \right)_K - \left( \hat{u}_m, \hat{T}[n]_m \right)_{\partial K} & = (\text{Grad} \hat{u}_m, C_m (I))_K - (\hat{u}_m, C_m (I) n_m)_{\partial K} + (\hat{u}_m, f)_K, \\
\left( \hat{F}_m, F^h_m \right)_K - \left( \hat{F}_m, \text{Grad} u^h_m \right)_K & + \left( \hat{F}_m n_m, (u^h_m - \mu^h_m) \right)_{\partial K} = (\hat{F}_m, I)_K, \\
\left( \hat{\mu}_m, \hat{T}[n]_m \right)_{\partial T^h_m \setminus \Gamma_{FSI}} & + \left( \hat{\mu}_m, u^h_m - u^h_m \right)_{\Gamma_{FSI}} = (\hat{\mu}_m, C_m (I) n_m)_{\partial T^h_m \setminus \Gamma_{FSI}} + (\hat{\mu}_m, g_{N_m})_{\partial \Omega_N}, \\
\forall \{ \hat{u}_m, \hat{F}_m, \hat{\mu}_m \} \in U^1_m \times F^1_m \times M^1_m,
\end{align*}
\]
where
\[
\hat{T}[n]_m := C_m \left( F^h_m \right) n_m - S_m \left( u^h_m - \mu^h_m \right),
\]
\[
S_m := \frac{\mu_m}{l_m} I.
\]

### 3.5 Monolithic fluid-structure interaction

The global FSI problem is a direct sum of Problems 1, 3, and 4 with coupling conditions.

**Problem 5** (Global FSI problem).

Find \( \{ L_h f, v_h f, p_h f, \psi_h f \} \in \mathcal{L}_h \times \mathcal{V}_h \times \mathcal{P}_h \times \mathcal{V}_h \), \( \{ u_h s, v_h s, E_h s, \psi_h s \} \in \mathcal{U}_s \times \mathcal{V}_s \times \mathcal{E}_s \times \mathcal{V}_s \), and \( \{ u^h_m, F^h_m, \mu^h_m \} \in \mathcal{U}_m \times \mathcal{F}_m \times \mathcal{M}_m \) such that Problems 1, 3, and 4 are simultaneously satisfied.

The globally defined system in Problem 5 is a fully coupled nonlinear system of equations. The Newton-Raphson procedure [24] is employed to solve these nonlinear equations.

### 4 Numerical results

Code-to-code comparison is considered against the benchmark given in [23] by Turek and Hron, with domain detailed in Fig. 1. All dimensions and properties used are from [23] for case “FSI2” and are reproduced in Table 1. The FSI benchmark simulates two-dimensional incompressible channel flow around a rigid cylinder with an attached nonlinearly elastic flag. The wake shed from the cylinder produces oscillations in the flag that self-excite and eventually reach a stationary oscillatory state. The initial conditions are zero

![Figure 1](image_url)

**Figure 1:** The domain used for the FSI benchmark proposed by [23]. The benchmark simulates two-dimensional channel flow over a rigid cylinder with a flexible flag attached.
Figure 2: A diagram of the (undeformed) mesh used for the HDG FSI simulation of the Turek and Hron benchmark [23]. There are 4217 fluid cells, indicated in black, and 147 solid cells, indicated in red.

Table 1: Turek and Hron [23] benchmark domain dimensions and material properties.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value [m]</th>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Channel length</td>
<td>$L$</td>
<td>2.5</td>
<td>$\rho_s \left[ \frac{kg}{m^3} \right]$</td>
<td>10000</td>
</tr>
<tr>
<td>Channel height</td>
<td>$H$</td>
<td>0.41</td>
<td>$\nu_s$</td>
<td>0.4</td>
</tr>
<tr>
<td>Cylinder center</td>
<td>$C$</td>
<td>(0.2, 0.2)</td>
<td>$E_s \left[ \frac{E6 \ kg}{m^2} \right]$</td>
<td>5.6</td>
</tr>
<tr>
<td>Cylinder radius</td>
<td>$r$</td>
<td>0.05</td>
<td>$\mu_l \left[ \frac{kg}{ms} \right]$</td>
<td>1000</td>
</tr>
<tr>
<td>Flag length</td>
<td>$l$</td>
<td>0.35</td>
<td>$\mu_f \left[ \frac{kg}{ms} \right]$</td>
<td>1</td>
</tr>
<tr>
<td>Flag height</td>
<td>$h$</td>
<td>0.02</td>
<td>$\bar{U} \left[ \frac{m}{s} \right]$</td>
<td>1</td>
</tr>
<tr>
<td>Reference point A</td>
<td>$A$</td>
<td>(0.6, 0.2)</td>
<td>$v_{f_in} \left[ \frac{m}{s} \right]$</td>
<td>$1.5 \Omega \frac{\psi(H - \psi)}{(H/2)^2}$</td>
</tr>
<tr>
<td>Reference point B</td>
<td>$B$</td>
<td>(0.15, 0.2)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

everywhere and the inlet velocity smoothly increases to $v_{f_in}$ over the first two seconds. After the first two seconds, the inlet velocity is constant. The top and bottom boundaries of the channel are subject to the no-slip condition, as is the perimeter of the rigid cylinder. The flag, attached to the cylinder, is allowed to move freely with the fluid flow, subject to the interface conditions specified in Eqs. (3.18), (3.19), and (3.20). The BDF2 timestepping method is used with a timestep size $\Delta t = 0.0025$ for all of the results in this section. The HDG FSI model and subproblems were implemented utilizing the deal.II finite element library [25, 26]. The mesh used for all the results in this section is presented in Fig. 2, which contains 4364 cells combined between the solid and fluid. Turek and Hron provided tip displacement data for point A, from Fig. 1, and the drag and lift calculated about the perimeter of the cylinder and attached flag. Fig. 3 contains plots for each of these metrics, with a comparison between the results from [3, 4], labeled “Old HDG FSI”, the results from the reduced DOF formulation presented in this work, labeled “Reduced DOF HDG FSI”, and the results from Turek and Hron’s benchmark [23], labeled “Turek FSI”. As can be seen in the figure, the results from [3, 4] and this work are indistinguishable from each other and, while there remains some discrepancy‡ between both sets of results.

‡We expect the discrepancy between the HDG FSI2 results and the Turek and Hron FSI2 results is caused by the choice of stabilization tensor $S$ for the system, as is discussed in more detail in [3, 4]. HDG stabilization for multi-physics systems is mostly unexplored in the literature and currently poorly understood. It remains an important area of future work.
HDG results and the Turek and Hron results, the important fact is that the new reduced DOF formulation documented in this work did not negatively impact the accuracy of the solution at all compared to the previous formulation, while substantially decreasing number of DOFs required for the calculation.

Table 2 lists the number of DOFs required by both the old and new HDG FSI formulations for the presented FSI2 results, as well as the percent reduction of DOFs with the formulation from this work over the old formulation. There is a total global DOF reduction of 17% and total local DOF reduction of 25%. This directly translates to improved computational efficiency, especially due to the global DOF reduction. It is worth noting that the solid global DOF reduction is 50%, so a problem with a larger solid domain relative to the fluid domain would experience even greater improvement in computational efficiency. Additionally, the case presented uses only quadratic tensor-product polyno-
Table 2: The number of DOFs for the solid domain and the fluid domain, for both the formulation from [3, 4] and this work’s. The fluid and mesh DOFs are both included in the fluid domain. The third set of columns shows the percent DOF reduction with the formulation presented in this work versus that from [3, 4].

<table>
<thead>
<tr>
<th></th>
<th>Old HDG FSI</th>
<th>Reduced DOF HDG FSI</th>
<th>Percent DOF reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Global DOFs</td>
<td>Local DOFs</td>
<td>Global DOFs</td>
</tr>
<tr>
<td>Fluid</td>
<td>108125</td>
<td>493389</td>
<td>90807</td>
</tr>
<tr>
<td>Solid</td>
<td>4152</td>
<td>10584</td>
<td>2076</td>
</tr>
<tr>
<td>Total</td>
<td>112277</td>
<td>503973</td>
<td>92883</td>
</tr>
</tbody>
</table>

For higher order elements, there will be a substantially increased DOF reduction (both locally and globally) due to the restriction of the mesh function spaces to only linear polynomials, as shown in Appendix A. Also, the fluid domain values from Table 2 directly agree with the arbitrary values from Table 4 for $d=k=2$.

In performing this numerical analysis, an additional, unexpected, benefit to the restricted mesh function spaces was discovered. The linear function spaces produced higher quality meshes and increased the stability of the system under large deformations compared to the equal-order mesh formulation. Problems with moving domains, such as FSI with the ALE Navier-Stokes equations, often suffer from not maintaining high mesh quality over time. If a mesh’s quality becomes so poor that it contains a negative volume, linear algebra solvers will be unable to produce a solution, resulting in a failed simulation. In this work, for this specific benchmark, the mesh’s material parameters are defined to make it radially less stiff outward from the tip of the flag. This disperses the mesh deformation throughout the mesh domain instead of localizing it about the FSI interface. With the linear mesh functions spaces cell quality was preserved at a higher level throughout the domain better than with the equal-order mesh function spaces.

4.1 Per-Element DOF-Reduction

The primary result from this work is the reduction in the DOFs per cell and face achieved while still maintaining the accuracy of the solution compared to the old method. The code-to-code comparison presented in the previous section demonstrates that the solution obtained by the new, less computationally expensive, method successfully predicts the Turek-Hron benchmark solution to the same accuracy as the old, more computationally expensive, method. In this section, a quantitative reduction in the number of DOFs per cell and cell face, for two and three dimensional elements, for polynomial orders $k=1-10$ are given.

Table 3 gives the number of DOFs for a fluid-ALE domain cell and cell face, computed by adapting the equations for the global number of unknowns given in Appendix A. In

---

§A poor quality mesh satisfies its boundary conditions but contains cells that are extremely distorted, such as cells with nearly parallel adjacent sides.
Table 3: Local DOFs for the fluid problem versus polynomial order and dimension. The new formulation results in a significant reduction in the number of DOFs.

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Local DOFs Per Fluid Cell</th>
<th>Global DOFs Per Fluid Face</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>OLD</td>
<td>NEW</td>
</tr>
<tr>
<td>Order ((k))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>52</td>
<td>52</td>
</tr>
<tr>
<td>2</td>
<td>117</td>
<td>87</td>
</tr>
<tr>
<td>3</td>
<td>208</td>
<td>136</td>
</tr>
<tr>
<td>4</td>
<td>325</td>
<td>199</td>
</tr>
<tr>
<td>5</td>
<td>468</td>
<td>276</td>
</tr>
<tr>
<td>6</td>
<td>637</td>
<td>367</td>
</tr>
<tr>
<td>7</td>
<td>832</td>
<td>472</td>
</tr>
<tr>
<td>8</td>
<td>1053</td>
<td>591</td>
</tr>
<tr>
<td>9</td>
<td>1300</td>
<td>724</td>
</tr>
<tr>
<td>10</td>
<td>1573</td>
<td>871</td>
</tr>
</tbody>
</table>

In each case, the number of DOFs is seen to reduce in both two and three dimensions as the finite element polynomial order \(k\) is increased. The effect of reformulating the problem is more pronounced in three dimensions than in two, indicating that more complex three-dimensional applications stand to gain significantly improved efficiency by using the new formulation.

The solid domain (elasticity) formulation has no reduction in the number of local per-cell DOFs, but does see a 50% reduction in the number of global DOFs, simply by removing one the global fields in the new formulation presented in this work.
5 Conclusion

This work presents a hybridizable discontinuous Galerkin finite element formulation for fluid structure interaction with substantially improved computational efficiency, in terms of a significant reduction in the number of global and local DOFs per cell, compared to the original HDG FSI formulation developed by Sheldon et al. [3, 4]. The resulting linearized system is smaller and does not increase the number of non-zeros for retained DOFs. Furthermore, the stability and accuracy of proposed formulation appear to be the same as the previous HDG FSI formulation.

The new formulation is shown to approach a 50% reduction in global DOF count for high-order simulations for an arbitrary FSI problem. For the specific numerical example of the Turek and Hron benchmark using only Q2 elements, the new formulation is shown to match the results of the original formulation with a negligible change in accuracy, while the number of total global DOFs is reduced by 17% and the number of total local DOFs is reduced by 25%. This work will form the basis for future studies, namely an empirical efficiency study to determine the efficiency of this method when applied to large problems.

Acknowledgments

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Appendix A: Fluid domain DOF reduction

This appendix demonstrates the fluid domain DOF reduction of the HDG FSI formulation presented in this work over the HDG FSI formulation previously presented in [3, 4]. These formulations are respectively referred to as “new” and “old” for brevity in the remainder of this appendix. Lagrange polynomials are used as basis functions in this example, but the concept extends to other basis functions with similar results. In the following equations and table, the variables \( d \) = dimension, \( k \) = polynomial order, \( f \) = unique faces in the fluid domain\(^8\), and \( c \) = cells in the fluid domain.

The number of fluid domain DOFs for an arbitrary HDG FSI are presented below for both the old and new formulations, along with the DOF reduction of the new formulation over the old.

\(^8\)The fluid domain includes both the fluid and mesh DOFs, even though only the mesh DOFs are reduced by the new formulation.
Local DOFs with old formulation: \( N_{lo} = c(2d^2 + 2d + 1)(k+1)^d \).

Local DOFs with new formulation: \( N_{ln} = c((k+1)^d + (d^2 + d)((k+1)^d + 2^d)) \).

Global DOFs with old formulation: \( N_{go} = 2fd(k+1)^{d-1} + c \).

Global DOFs with new formulation: \( N_{gn} = fd((k+1)^{d-1} + 2^{d-1}) + c \).

Local DOF reduction from old to new: \( R_l = 1 - \frac{N_{ln}}{N_{lo}} = \frac{(d^2 + d) \left(1 - 2^d (k+1)^{-d}\right)}{2d^2 + 2d + 1} \).

Global DOF reduction from old to new: \( R_g = 1 - \frac{N_{gn}}{N_{go}} = \frac{1}{2} \frac{(k+1)(c+2^d df)}{ck+c+2df(k+1)^d} \).

Applying this to specific values of \( d \) and \( k \) yields the results in Table 4, which show that as polynomial order increases, the local and global percent DOF reduction of the new formulation over the old approaches approximately 50%.

<table>
<thead>
<tr>
<th>( k )</th>
<th>( R_l ) for ( d = 2 )</th>
<th>( R_l ) for ( d = 3 )</th>
<th>( R_g ) for ( d = 2 )</th>
<th>( R_g ) for ( d = 3 )</th>
<th>( \lim_{k \to \infty} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k = 2 )</td>
<td>25.64%</td>
<td>34.62%</td>
<td>38.77%</td>
<td>41.03%</td>
<td>46.15%</td>
</tr>
<tr>
<td>( k = 3 )</td>
<td>33.78%</td>
<td>42.00%</td>
<td>44.93%</td>
<td>46.22%</td>
<td>48.00%</td>
</tr>
<tr>
<td>( k = 4 )</td>
<td>16.00%</td>
<td>24.24%</td>
<td>29.27%</td>
<td>32.65%</td>
<td>50.00%</td>
</tr>
<tr>
<td>( k = 5 )</td>
<td>27.61%</td>
<td>37.37%</td>
<td>41.91%</td>
<td>44.38%</td>
<td>50.00%</td>
</tr>
</tbody>
</table>

Table 4: The fluid domain DOF reduction from the new HDG FSI formulation over the old formulation; \( f = cd \) for the finite \( R_g \) values. The limit as \( k \to \infty \) is exact.

References


URL http://dx.doi.org/10.1002/9781118483565.fmatter
URL http://www.dealii.org