A Posteriori Error Estimates of Discontinuous Streamline Diffusion Methods for Transport Equations

Juan Sun¹, Zhaojie Zhou² and Huipo Liu³,*

¹ Key Lab. of Machine Learning and Computational Intelligence, College of Mathematics and Information Science, Hebei University, Baoding, 071002, China.
² School of Mathematics and Statistics, Shandong Normal University, Ji’nan, 250014, China.
³ Institute of Applied Physics and Computational Mathematics, Beijing, 100094, China.

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Abstract. Residual-based posteriori error estimates for discontinuous streamline diffusion methods for transport equations are studied in this paper. Computable upper bounds of the errors are measured based on mesh-dependent energy norm and negative norm. The estimates obtained are locally efficient, and thus suitable for adaptive mesh refinement applications. Numerical experiments are provided to illustrate underlying features of the estimators.

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1 Introduction

A posteriori error estimates and corresponding adaptive computation have been an active research field in recent years, especially for elliptic and parabolic equations. Since the pioneering work of Babuška and Rheinboldt [1], a large number of work were devoted to develop a posteriori error estimates and adaptive algorithms. Residual type a posteriori error estimates have been developed, e.g., in [1–8]. Recovery type a posteriori error estimates have been established in [9–13]. For more references about a posteriori error estimate one can refer to [14–18].

As we know the standard Galerkin finite element approximation of transport equation often leads to interior or boundary layer where the gradient of solution changes

*Corresponding author. Email addresses: jsun@hbu.edu.cn (J. Sun), zhouzhaojie@sdnu.edu.cn (Z. Zhou), liuhuipo@amss.ac.cn (H. Liu)

rapidly. Therefore, various stabilization methods have been developed, e.g., discontinuous Galerkin methods [19–21], streamline diffusion methods [22, 23], finite volume methods [24], interior penalty finite element methods [25] and discontinuous streamline diffusion method [26]. Compared with elliptic and parabolic equation a posteriori error estimate for the transport equation is far less developed in the literature. This is mainly due to the fact that the elliptic problem has some smoothing and stability properties. A posteriori error estimate is mainly based on the stability of the problem on the continuous level. The transport equation’s stability on the continuous level is not as good as sample elliptic problems. A posteriori error estimates for discontinuous Galerkin methods and streamline diffusion methods of transport equations based on dual argument and nonstandard norm can be found, e.g., in [27–29]. Based on the saturation assumption and interpolation estimate between discrete space a posteriori error estimates for the transport equation have been presented in [25, 30]

In this paper we mainly focus on developing a posteriori error estimate for discontinuous streamline diffusion approximation of transport equations. This method not only keeps the advantage of the upwind approach, but also improves the stability of discontinuous Galerkin method. The stability and a priori error estimate in certain norms of the discontinuous streamline diffusion method can be found in [26, 31]. Residual type a posteriori error estimates in mesh dependent norm and $H^{-1}$ norm for discontinuous streamline diffusion approximation of transport equations are derived. Numerical examples are given to illustrate the theoretical findings.

The paper is organized as follows. In the next section we briefly recall the discontinuous streamline diffusion approximation of transport equation including stability estimate and a priori error estimate. In Section 3 we provide two residual type a posteriori error estimates of discontinuous streamline diffusion methods in mesh dependent norm and $H^{-1}$-norm. In Section 4 we provide several numerical tests which support our theory. Finally, in Section 5 we summarize the work presented in this paper.

Throughout this paper, some standard notations are used for Sobolev spaces, and the corresponding semi-norms and norms [32]. Moreover, the letter C denotes a generic constant which may stand for different values at its different occurrences and is independent of the mesh parameters.

2 Preliminaries

2.1 The discontinuous streamline diffusion method

Let $\Omega$ be a polygon in $\mathbb{R}^2$ with a boundary $\Gamma$. Suppose that $a = (a_1, a_2)$ is a vector function defined on $\Omega$ with $a_1, a_2 \in W^{1}_{\infty}(\Omega)$, and consider the following subsets of $\Gamma$:

$$\Gamma_- = \{x \in \Gamma : a \cdot n(x) < 0\}, \quad \Gamma_+ = \{x \in \Gamma : a \cdot n(x) \geq 0\},$$

where $n(x)$ is the unit outward normal at the point $x \in \partial \Omega$. The sets $\Gamma_-$ and $\Gamma_+$ are referred as the inflow and outflow boundary, respectively. We consider the following transport
problem: finding \(u \in H(L, \Omega)\) such that
\[
Lu = a \cdot \nabla u + bu = f \quad \text{in} \quad \Omega,
\]
\[
u = g \quad \text{on} \quad \Gamma_-,
\]
where \(H(L, \Omega) = \{v \in L^2(\Omega) : Lv \in L^2(\Omega)\}\) denotes the space of the partial differential operator \(L\) in \(L^2(\Omega)\), \(b \geq 0 \in L^\infty(\Omega)\), \(f \in L^2(\Omega)\), and \(g \in L^2(\Gamma_-)\). We adopt the following standard hypothesis
\[
\inf_{x \in \Omega} \left( b(x) - \frac{1}{2} \nabla \cdot a(x) \right) \geq \gamma_0 > 0.
\]

In neutron physics, \(u\) is the so-called neutron flux, \(a\) is a given unit vector, \(b\) is related to the probability of interaction between the neutrons and surrounding nuclei in \(\Omega\), and \(f\) is a source term. We note that assumption (2.2) ensure the uniqueness of a solution \(u \in H(L, \Omega)\) to (2.1), see [33]. For \(u \in H^1(\Omega)\), the following stability estimate [34] holds
\[
\|a \cdot \nabla u\|_{0, \Omega} + \|u\|_{0, \Omega} \leq C(\|f\|_{0, \Omega} + \|g\|_{0, \Gamma_-}).
\]

Let \(T_h\) be a family of shape regular triangulations of \(\bar{\Omega}\) into triangles \(\tau\). Let \(h_\tau\) denote the diameter of \(\tau\) and \(\rho_\tau\) denote the diameter of the biggest ball included in \(\tau\), and let \(h = \max_\tau h_\tau\). As usual, the families of meshes satisfy the classical assumption of shape regularity (see [35]); namely, there is a constant \(C > 0\) independent of \(h\) such that
\[
\frac{h_\tau}{\rho_\tau} \leq C, \quad \forall \tau \in T_h.
\]

The assumption (2.4) admits locally adapted meshes with non-degenerating elements. For any element \(\tau \in T_h\), we denote by \(\partial \tau\) the edge of \(\tau\). Then, the inflow and outflow parts of \(\partial \tau\) are defined by
\[
\partial \tau_- = \{x \in \partial \tau : a(x) \cdot n(x) < 0\}, \quad \partial \tau_+ = \{x \in \partial \tau : a(x) \cdot n(x) \geq 0\},
\]
where \(n(x)\) denotes the unit outward normal vector to \(\partial \tau\) at \(x \in \partial \tau\). We partition the domain \(\Omega\) sufficiently fine, such that \(a(x) \cdot n(x)\) does not change sign on every edge of \(\tau\). Thus, every edge is either inflow or outflow, respectively.

To define the discontinuous streamline scheme we also need to introduce some notations. For a nonnegative integer \(k\), we denote by \(P_k(\tau)\) the set of polynomials of total degree \(k\) on \(\tau\). For a given function \(v \in P_k(\tau)\), we define the left and right hand limits \(v^-\) and \(v^+\) by
\[

v^-(x) = \lim_{s \to 0^-} v(x + sa), \quad v^+(x) = \lim_{s \to 0^+} v(x + sa),
\]
for $x \in \partial \tau$. Let $\Gamma_h$ denote the set of edges of $\mathcal{T}_h$, and define the jump $[v]$ across $\partial \tau \in \Gamma_h$ by

$$
[v] = \begin{cases} 
    v^+ - v^- & \text{if } \partial \tau \in \Gamma_0, \\
    v^+ \equiv v & \text{if } \partial \tau \subset \Gamma_-, \\
    v^- \equiv v & \text{if } \partial \tau \subset \Gamma_+,
\end{cases}
$$

(2.5)

where $\Gamma_0$ denotes the set of interior edges in $\mathcal{T}_h$. The $L^2$-inner product and norm in $L^2(\tau)$ and $L^2(\partial \tau)$ are defined by

$$
(w,v)_{\tau} = \int_{\tau} wvdx, \quad \|w\|_{0,\tau}^2 = (w,w)_{\tau}, \quad \forall w,v \in L^2(\tau),$$

$$
(w,v)_{\partial \tau} = \int_{\partial \tau} wvd\sigma, \quad \|w\|_{0,\partial \tau}^2 = (w,w)_{\partial \tau}, \quad \forall w,v \in L^2(\partial \tau).
$$

In the following, we introduce a finite element space on $\mathcal{T}_h$

$$
V_h = \left\{ v \in L^2(\Omega) : v|_{\tau} \in P_k(\tau) \quad \forall \tau \in \mathcal{T}_h \right\},
$$

(2.6)

and a piecewise smooth function space on $\mathcal{T}_h$

$$
H^1(\mathcal{T}_h) = \left\{ v \in L^2(\Omega) : v|_{\tau} \in H^1(\tau) \quad \forall \tau \in \mathcal{T}_h \right\}.
$$

(2.7)

Obviously, we have $V_h \subset H^1(\mathcal{T}_h)$. Then the discontinuous streamline diffusion approximation of (2.1) is to find $u_h \in V_h$ such that

$$
a_h(u_h,v_h) = l_h(v_h) \quad \forall v_h \in V_h.
$$

(2.8)

Here the bilinear form $a_h$ and the linear form $l_h$ are given by

$$
a_h(u_h,v_h) = \sum_{\tau \in \mathcal{T}_h} (\mathcal{L} u_h,v_h + \delta a \cdot \nabla v_h)_\tau + \sum_{\partial \tau \in \Gamma_0} \int_{\partial \tau} (1+\delta b)[u_h]_h^+ |a \cdot n| ds
$$

$$
+ \int_{\partial \tau} (1+\delta b)u_h v_h |a \cdot n| ds,
$$

$$
l_h(v_h) = \sum_{\tau \in \mathcal{T}_h} (f,v_h + \delta a \cdot \nabla v_h)_\tau + \int_{\partial \tau} (1+\delta b)g v_h |a \cdot n| ds,
$$

where $\delta > 0$ is artificial diffusion viscosity parameter. In general, we choose $\delta = C_\delta h$. When $\delta = 0$, the discontinuous streamline diffusion method will reduce to the discontinuous Galerkin method.

Let $u \in H^1(\Omega)$ and $u_h \in V_h$ be the solutions to (2.1) and (2.8), respectively. We have

$$
a_h(u-u_h,v_h) = 0 \quad \forall v_h \in V_h.
$$

(2.9)

For the following analysis, we need the trace inequality (see [35]).

**Lemma 2.1.** Let $v|_{\tau} \in H^1(\tau)$; then

$$
\|v\|_{0,\partial \tau} \leq C(h_{\tau}^{-\frac{1}{2}} |v|_{0,\tau} + h_{\tau}^{\frac{1}{2}} \|\nabla v\|_{0,\tau}).
$$

(2.10)
2.2 Stability analysis

Let us first study the coerciveness of \( a_h \) on \( H^1(T_h) \), which guarantees the unique solvability of the problem (2.8). For the error analysis, we introduce the mesh-dependent stability norm

\[
|||w|||_h^2 = \sum_{\tau \in T_h} \left( \delta \|a \cdot \nabla w\|_{0, \tau}^2 + \frac{\gamma_0}{2} \|w\|_{0, \tau}^2 \right) + \frac{1}{2} \sum_{\partial \tau \in \Gamma_0} \int_{\partial \tau} (1 + \delta b) |w|^2 |a \cdot n| ds \\
+ \frac{1}{2} \int_{\Gamma} (1 + \delta b) |w|^2 |a \cdot n| ds.
\]

For completeness, we introduce a stability estimate for the discontinuous streamline diffusion method (2.8) by using same method in [26].

**Lemma 2.2.** Let \( H^1(T_h) \) be the piecewise smooth function space defined by (2.7). Then the discrete bilinear form \( a_h \) is coercive on \( H^1(T_h) \), i.e.,

\[
a_h(w, w) \geq |||w|||_h^2 \quad \forall w \in H^1(T_h).
\]

**Proof.** Noting the definition of \( a_h \), we get

\[
a_h(w, w) = \sum_{\tau \in T_h} \left( \mathcal{L} w, w + \delta a \cdot \nabla w \right)_\tau \\
+ \sum_{\partial \tau \in \Gamma_0} \int_{\partial \tau} (1 + \delta b) |w|^2 |a \cdot n| ds + \int_{\Gamma} (1 + \delta b) |w|^2 |a \cdot n| ds \\
= \sum_{\tau \in T_h} \left( \delta \|a \cdot \nabla w\|_{0, \tau}^2 + (bw, w)_\tau + (w, (1 + \delta b) a \cdot \nabla w)_\tau \right) \\
+ \sum_{\partial \tau \in \Gamma_0} \int_{\partial \tau} (1 + \delta b) |w|^2 |a \cdot n| ds + \int_{\Gamma} (1 + \delta b) |w|^2 |a \cdot n| ds.
\]

By Green’s formula and (2.2) we have

\[
(bw, w)_\tau + (w, (1 + \delta b) a \cdot \nabla w)_\tau \\
= ((b - \frac{1}{2} \nabla \cdot a) w, w)_\tau - \delta \int_{\partial \tau} ((a \cdot \nabla b + b \nabla \cdot a) w, w)_\tau + \frac{1}{2} \int_{\partial \tau} (1 + \delta b) |w|^2 a \cdot n ds \\
\geq \gamma_0 \|w\|_{0, \tau}^2 - \delta \left( \|a \cdot \nabla b + b \nabla \cdot a\|_{\infty, \tau} \right) + \frac{1}{2} \left( \int_{\partial \tau} (1 + \delta b) |w|^2 a \cdot n ds \\
- \int_{\partial \tau} (1 + \delta b) |w|^2 a \cdot n ds \right).
\]

Choosing \( 0 < C_{\delta} \leq C_0 \) and \( 0 < h \leq h_0 \) then gives

\[
\frac{C_{\delta} h}{2} \|a \cdot \nabla b + b \nabla \cdot a\|_{\infty, \tau} \leq \frac{\gamma_0}{2}.
\]
Inserting (2.14) into (2.13) yields
\[
\sum_{\tau \in T_h} \left( (bw, w)_\tau + (w, (1 + \delta b) \mathbf{a} \cdot \nabla w)_\tau \right)
\geq \sum_{\tau \in T_h} \frac{\gamma_0}{2} \|w\|_{0, \tau}^2 + \frac{1}{2} \sum_{\tau \in T_h} \left( \int_{\partial \tau_i} (1 + \delta b)(w^-)^2 |\mathbf{a} \cdot \mathbf{n}| ds 
- \int_{\partial \tau_i} (1 + \delta b)(w^+)^2 |\mathbf{a} \cdot \mathbf{n}| ds \right).
\] (2.15)

Obviously we have that
\[
\sum_{\tau \in T_h} \int_{\partial \tau} (1 + \delta b)(w^-)^2 |\mathbf{a} \cdot \mathbf{n}| ds
= \sum_{\partial \tau \in \Gamma_0} \int_{\partial \tau} (1 + \delta b)(w^-)^2 |\mathbf{a} \cdot \mathbf{n}| ds + \int_{\Gamma_+} (1 + \delta b)(w) |\mathbf{a} \cdot \mathbf{n}| ds + \int_{\Gamma_-} (1 + \delta b)(w) |\mathbf{a} \cdot \mathbf{n}| ds,
\] (2.16)

and
\[
\sum_{\tau \in T_h} \int_{\partial \tau} (1 + \delta b)(w^+)^2 |\mathbf{a} \cdot \mathbf{n}| ds
= \sum_{\partial \tau \in \Gamma_0} \int_{\partial \tau} (1 + \delta b)(w^+)^2 |\mathbf{a} \cdot \mathbf{n}| ds + \int_{\Gamma_+} (1 + \delta b)(w) |\mathbf{a} \cdot \mathbf{n}| ds + \int_{\Gamma_-} (1 + \delta b)(w) |\mathbf{a} \cdot \mathbf{n}| ds.
\] (2.17)

Inserting (2.15)-(2.17) into (2.12), it follows from (2.5) that
\[
a_h(w, w) \geq \sum_{\tau \in T_0} \left( \delta \|\mathbf{a} \cdot \nabla w\|_{0, \tau}^2 + \frac{\gamma_0}{2} \|w\|_{0, \tau}^2 \right) + \frac{1}{2} \sum_{\partial \tau \in \Gamma_0} \int_{\partial \tau} (1 + \delta b)|w|^2 |\mathbf{a} \cdot \mathbf{n}| ds
\]
\[
+ \frac{1}{2} \int_{\Gamma} (1 + \delta b)(w)^2 |\mathbf{a} \cdot \mathbf{n}| ds = |||w|||^2_{h},
\] (2.18)

which completes the proof of the lemma. \(\square\)

Using Lemma 2.2 and the approximation properties of the space \(V_h\), the following a priori error estimate is obtained. Let \(u \in H^{1+r}(\Omega)\) be the solution of (2.1) and \(u_h \in V_h\) be the solution of (2.8). For \(\delta = C \delta_h > 0\), the following a priori error estimate holds:
\[
||| u - u_h |||_h \leq C_h^{\min\{r, k\}} + 1/2 ||| u |||_{1+r, \Omega}.
\] (2.19)

For more details of a priori error estimate, the reader is referred to the methods of Refs. [20, 26]. In this paper, we will concentrate our discussion on a posteriori error analysis.
3 A posteriori error analysis

The objective of this section is to develop residual type a posteriori error estimates for discontinuous streamline diffusion approximation of transport equations in mesh-dependent norm and negative norm, respectively.

3.1 A posteriori error estimate in mesh-dependent norm

**Theorem 3.1.** Let \( u \in H^1(\Omega) \) and \( u_h \in V_h \) be the exact solution and the discontinuous streamline diffusion method solution of problems (2.1) and (2.8), respectively. Then the following a posteriori error estimate holds:

\[
|||u - u_h|||^2_h \leq C_1(f, g)(\eta_{\Omega}(u_h) + \eta_{\Gamma_0}(u_h) + \eta_{\Gamma_-}(u_h))^\frac{1}{2},
\]

(3.1)

where

\[
\eta_{\Omega}(u_h) = \left( \sum_{\tau \in T_h} \| f - L u_h \|_{0, \tau}^2 \right)^{\frac{1}{2}},
\]

\[
\eta_{\Gamma_0}(u_h) = \left( \sum_{\partial \tau \in \Gamma_0} h^{-1}_\tau \| (1 + \delta) [u_h] a \cdot n \|_{0, \partial \tau_0}^2 \right)^{\frac{1}{2}},
\]

\[
\eta_{\Gamma_-}(u_h) = \left( \sum_{\partial \tau \in \Gamma_-} \| (1 + \delta) (g - u_h) a \cdot n \|_{0, \partial \tau_-}^2 \right)^{\frac{1}{2}},
\]

and \( C_1(f, g) \) is a bounded constant and dependent of \( f \) and \( g \).

**Proof.** Using (2.11) and orthogonality equation (2.9), we obtain

\[
|||u - u_h|||^2_h \leq a_h(u - u_h, u - u_h) = a_h(u - u_h, u)
\]

\[
= \sum_{\tau \in T_h} \left( L(u - u_h), u + \delta a \cdot \nabla u \right)_\tau + \sum_{\partial \tau \in \Gamma_0} \int_{\partial \tau_0} (1 + \delta) [u_h] a \cdot n \| ds
\]

\[
+ \int_{\Gamma_-} (1 + \delta) (g - u_h) a \cdot n \| ds
\]

\[
= \sum_{\tau \in T_h} \left( f - Lu_h, u + \delta a \cdot \nabla u \right)_\tau - \sum_{\partial \tau \in \Gamma_0} \int_{\partial \tau_0} (1 + \delta) [u_h] a \cdot n \| ds
\]

\[
+ \int_{\Gamma_-} (1 + \delta) (g - u_h) a \cdot n \| ds
\]

\[
= I + II + III.
\]

(3.2)

In the following we are going to estimate the terms \( I, II \) and \( III \), respectively. Firstly, using the Cauchy-Schwarz inequality and stability estimate (2.3) we have

\[
I = \sum_{\tau \in T_h} \left( f - Lu_h, u + \delta \tau a \cdot \nabla u \right)_\tau \leq \sum_{\tau \in T_h} \| f - Lu_h \|_\tau \| u + \delta \tau a \cdot \nabla u \|_\tau
\]
For the inflow boundary term, using the Cauchy-Schwarz inequality leads to

\[
W \leq \left( \sum_{T \in T_h} \left\| f - \mathcal{L} u_h \right\|_{0,T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} \left\| u + \delta_T a \cdot \nabla u \right\|_{0,T}^2 \right)^{\frac{1}{2}}
\]

\[
= \left( \sum_{T \in T_h} \left\| f - \mathcal{L} u_h \right\|_{0,T}^2 \right)^{\frac{1}{2}} \left\| u + \delta_T a \cdot \nabla u \right\|_{0,\Omega}
\]

\[
\leq \left( \sum_{T \in T_h} \left\| f - \mathcal{L} u_h \right\|_{0,T}^2 \right)^{\frac{1}{2}} \left( \left\| u \right\|_{0,\Omega} + \left\| \delta_T a \cdot \nabla u \right\|_{0,\Omega} \right)
\]

\[
\leq C \left( \sum_{T \in T_h} \left\| f - \mathcal{L} u_h \right\|_{0,T}^2 \right)^{\frac{1}{2}} (\left\| f \right\|_{0,\Omega} + \left\| g \right\|_{0,\tau_-}).
\] (3.3)

For the weakly consistent jump term, using the Cauchy-Schwarz inequality, stability estimate (2.3) and trace inequality (2.10) we have

\[
II = \sum_{\partial \Omega \in T_0} \int_{\partial \Omega} (1 + \delta b)[u_h]a \cdot n |ds| \leq \sum_{\partial \Omega \in T_0} \left\| (1 + \delta b)[u_h]a \cdot n \right\|_{0,\partial \tau_-} \left\| u \right\|_{0,\partial \tau_-}
\]

\[
\leq \left( \sum_{\partial \tau \in T_0} \left\| (1 + \delta b)[u_h]a \cdot n \right\|_{0,\partial \tau_-}^2 \right)^{\frac{1}{2}} \left( \sum_{\partial \tau \in T_0} \left\| u \right\|_{0,\partial \tau_-}^2 \right)^{\frac{1}{2}}
\]

\[
\leq C \left( \sum_{\partial \tau \in T_0} h^{-1}_\tau \left\| (1 + \delta b)[u_h]a \cdot n \right\|_{0,\partial \tau_-}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} h^{-1}_\tau \left\| u \right\|_{0,\tau} + h^2 \left\| \nabla u \right\|_{0,\tau} \right)^{\frac{1}{2}}
\]

\[
\leq C \left( \sum_{\partial \tau \in T_0} h^{-1}_\tau \left\| (1 + \delta b)[u_h]a \cdot n \right\|_{0,\partial \tau_-}^2 \right)^{\frac{1}{2}} \left( \left\| u \right\|_{0,\Omega} + h^2 \left\| \nabla u \right\|_{0,\Omega} \right)
\]

\[
\leq C \left( \sum_{\partial \tau \in T_0} h^{-1}_\tau \left\| (1 + \delta b)[u_h]a \cdot n \right\|_{0,\partial \tau_-}^2 \right)^{\frac{1}{2}} \left( \left\| f \right\|_{0,\Omega} + \left\| g \right\|_{0,\tau_-} \right). \] (3.4)

For the inflow boundary term, using the Cauchy-Schwarz inequality leads to

\[
III = \int_{\Gamma_-} (1 + \delta b)(g - u_h)g a \cdot n |ds| \leq \left\| (1 + \delta b)(g - u_h) a \cdot n \right\|_{0,\tau_-} \left\| g \right\|_{0,\tau_-}
\]

\[
\leq C \left( \sum_{\partial \tau \in T_0} \left\| (1 + \delta b)(g - u_h) a \cdot n \right\|_{0,\partial \tau_-}^2 \right)^{\frac{1}{2}} \left( \left\| f \right\|_{0,\Omega} + \left\| g \right\|_{0,\tau_-} \right). \] (3.5)

Inserting (3.3)-(3.5) into (3.2) yields

\[
\left\| u - u_h \right\|_{0,T}^2 \leq C(\eta_{\Omega}(u_h) + \eta_{\Omega_0}(u_h) + \eta_{\tau_-}(u_h))(\left\| f \right\|_{0,\Omega} + \left\| g \right\|_{0,\tau_-})
\]

\[
\leq C(f, g)(\eta_{\Omega}^2(u_h) + \eta_{\Omega_0}^2(u_h) + \eta_{\tau_-}^2(u_h))^2. \]

We complete the proof of the lemma.

\[\square\]
3.2 A posteriori error estimate in negative norm

In this section we will derive a posteriori error estimate in negative norm for above discontinuous streamline diffusion scheme. For this purpose we introduce an auxiliary problem:

\[ \mathcal{L}^* z \equiv -a \cdot \nabla z + (b - \nabla \cdot a) z = \psi \ \text{in} \ \Omega, \]

\[ z = 0 \ \text{on} \ \Gamma_+. \]

For \( \psi \in H^1_0(\Omega) \) the following regularity estimate holds

\[ \|z\|_{1,\Omega} \leq C \|\psi\|_{1,\Omega}, \quad (3.6) \]

which can be viewed as a special case of the general differentiability theorem of Rauch’s work in [36]. We define the negative Sobolev norm \( \|\cdot\|_{-1,\Omega} \) in the usual way:

\[ \|w\|_{-1,\Omega} = \sup_{\psi \in H^1_0(\Omega)} \frac{(w,\psi)}{\|\psi\|_{1,\Omega}}. \]

To derive a posteriori error estimate we also need to introduce an interpolation operator from \( H^1(\tau) \) to piecewise constant space. Following [35, 37] we have the following lemma.

**Lemma 3.1.** Let \( \pi_h \) be such that

\[ \pi_h p|_\tau = \int_{\tau} \frac{p}{|\tau|}, \quad \forall p \in H^1(\tau), \ \tau \in T_h, \]

where \( |\tau| \) is the measure of \( \tau \). Then for \( p \in H^1(\Omega) \) we have

\[ \|p - \pi_h p\|_{0,\tau} \leq C h_\tau |\nabla p|_{0,\tau}, \quad (3.7) \]

\[ \|\pi_h p\|_{0,\tau} \leq \|p\|_{0,\tau}. \quad (3.8) \]

**Theorem 3.2.** Let \( u \in H^1(\Omega) \) and \( u_h \in V_h \) be the exact solution and the discontinuous streamline diffusion method solution of problems (2.1) and (2.8), respectively. Then the following a posteriori error estimate holds:

\[ \|u - u_h\|_{-1,\Omega} \leq C (E^2_\Omega(u_h) + E^2_{\Gamma_0}(u_h) + E^2_{\Gamma_-}(u_h))^{1/2}, \quad (3.9) \]

where

\[ E_{\Omega}(u_h) = \left( \sum_{\tau \in T_h} \|h_\tau (f - \mathcal{L} u_h)\|_{0,\tau}^2 \right)^{1/2}, \]

\[ E_{\Gamma_0}(u_h) = \left( \sum_{\partial \tau \in \Gamma_0} \|h^2_\tau [u_h] a \cdot n\|_{0,\partial \tau}^2 \right)^{1/2}, \]

\[ E_{\Gamma_-}(u_h) = \left( \sum_{\partial \tau \in \Gamma_-} \|h^2_\tau (g - u_h) a \cdot n\|_{0,\partial \tau}^2 \right)^{1/2}. \]
Proof. For $\psi \in H_0^1(\Omega)$ by using the adjoint problem and integration formula by parts, we have
\[
(u - u_h, \psi) = (u - u_h, L^* z) = \sum_{\tau \in T_h} \left((L(u - u_h), z)_\tau - \int_{\partial \tau} z(u - u_h) a \cdot n \, ds\right)
\]
\[
= \sum_{\tau \in T_h} (f - Lu_h, z)_\tau + \sum_{\partial \tau \in 1_0} \int_{\partial \tau} z[u - u_h] a \cdot n |ds + \int_{\Gamma_r} z(g - u_h) a \cdot n |ds.
\]
By the orthogonality property (2.9) and (3.10), we deduce that, for any $z_h \in V_h$,
\[
(u - u_h, \psi) = \sum_{\tau \in T_h} \left((f - Lu_h, z - z_h)_\tau - (f - Lu_h, \delta a \cdot \nabla z_h)_\tau\right)
\]
\[
+ \sum_{\partial \tau \in 1_0} \int_{\partial \tau} (z - z_h^+ - \delta bz^+_h)[u - u_h] a \cdot n |ds + \int_{\Gamma_r} (z - z_h - \delta bz_h)(g - u_h) a \cdot n |ds
\]
\[
\equiv I + II + III.
\]
In the following analysis we choose $z_h = \tau_h z$. Noting that $a \cdot \nabla z_h = 0$, it follows from the Cauchy-Schwarz inequality and (3.7) that
\[
I = \sum_{\tau \in T_h} \left((f - Lu_h, z - z_h)_\tau - (f - Lu_h, \delta a \cdot \nabla z_h)_\tau\right)
\]
\[
\leq \sum_{\tau \in T_h} \|h_T(f - Lu_h)\|_{0,\tau} \|h^{-1}_T(z - z_h)\|_{0,\tau}
\]
\[
\leq \left(\sum_{\tau \in T_h} \|h_T(f - Lu_h)\|_{0,\tau}^2\right)^{\frac{1}{2}} \left(\sum_{\tau \in T_h} \|h^{-1}_T(z - z_h)\|_{0,\tau}^2\right)^{\frac{1}{2}}
\]
\[
\leq C \left(\sum_{\tau \in T_h} \|h_T(f - Lu_h)\|_{0,\tau}^2\right)^{\frac{1}{2}} \|z\|_{1,\Omega}.
\]
By Cauchy-Schwarz inequality we derive
\[
II = \sum_{\partial \tau \in 1_0} \int_{\partial \tau} (z - z_h^+ - \delta bz_h^+)[u - u_h] a \cdot n |ds
\]
\[
= \sum_{\partial \tau \in 1_0} \int_{\partial \tau} (z - z_h^+ - \delta bz_h^+) [-u_h] a \cdot n |ds
\]
\[
\leq \sum_{\partial \tau \in 1_0} \|h_T^{\frac{1}{2}} [u_h] a \cdot n \|_{0,\partial \tau} \|h^{-\frac{1}{2}}_T (z - z_h^+ - \delta bz_h^+)\|_{0,\partial \tau}
\]
\[
\leq \left(\sum_{\partial \tau \in 1_0} \|h_T^{\frac{1}{2}} [u_h] a \cdot n \|_{0,\partial \tau}^2\right)^{\frac{1}{2}} \left(\sum_{\partial \tau \in 1_0} \|h^{-\frac{1}{2}}_T (z - z_h^+ - \delta bz_h^+)\|_{0,\partial \tau}^2\right)^{\frac{1}{2}}.
\]
Collecting (2.10), (3.7), (3.8) gives
\[
\left( \sum_{\partial \tau \in \Gamma_0} \| h^{-\frac{1}{2}}(z - z_h^+ - \delta b z_h^+) \|^2_{\partial \tau} \right)^{\frac{1}{2}} \leq \sum_{\partial \tau \in \Gamma_0} \| h^{-\frac{1}{2}}(z - z_h^+ - \delta b z_h^+) \|_{0, \partial \tau}.
\]
\[
\leq \sum_{\tau \in T_h} \left( h^{-\frac{1}{2}} \| z - z_h^+ \|_{0, \partial \tau} + h^{\frac{1}{2}} \| \delta b z_h^+ \|_{0, \partial \tau} \right)
\]
\[
\leq \sum_{\tau \in T_h} \left( h^{-\frac{1}{2}} \| z - z_h^+ \|_{0, \tau} + \| \nabla (z - z_h) \|_{0, \tau} + C_\delta h^{\frac{1}{2}} \| b \|_{\infty, \tau} (h^{\frac{1}{2}} \| z_h \|_{0, \tau} + h^{\frac{1}{2}} \| \nabla z_h \|_{0, \tau}) \right)
\]
\[
\leq \sum_{\tau \in T_h} \left( C \| \nabla z \|_{0, \tau} + C_\delta \| b \|_{\infty, \tau} \| z_h \|_{0, \tau} \right) \leq C \| z \|_{1, \Omega}.
\]
(3.14)

Inserting (3.14) into (3.13) yields
\[
II \leq \left( \sum_{\partial \tau \in \Gamma_0} \| h^{\frac{1}{2}}(u_h) \| a \cdot n \|_{0, \partial \tau} \right)^{\frac{1}{2}} \| z \|_{1, \Omega}.
\]
(3.15)

Similarly, we can deduce
\[
III = \int_{\Gamma_-} (z - z_h - \delta b z_h)(g - u_h) a \cdot n \ ds
\]
\[
\leq \sum_{\partial \tau \in \Gamma_-} \| h^{\frac{1}{2}}(g - u_h) \| a \cdot n \|_{0, \partial \tau} \| h^{-\frac{1}{2}}(z - z_h^+ - \delta b z_h^+) \|_{0, \partial \tau}.
\]
\[
\leq \left( \sum_{\partial \tau \in \Gamma_-} \| h^{\frac{1}{2}}(g - u_h) \| a \cdot n \|_{0, \partial \tau} \right)^{\frac{1}{2}} \left( \sum_{\partial \tau \in \Gamma_-} \| h^{-\frac{1}{2}}(z - z_h^+ - \delta b z_h^+) \|_{0, \partial \tau} \right)^{\frac{1}{2}}
\]
\[
\leq C \left( \sum_{\partial \tau \in \Gamma_-} \| h^{\frac{1}{2}}(g - u_h) \| a \cdot n \|_{0, \partial \tau} \right)^{\frac{1}{2}} \| z \|_{1, \Omega}.
\]
(3.16)

Therefore, it follows from (3.11), (3.15) and (3.16) that
\[
\left( u - u_h, \psi \right) \leq C(E_{\Omega}(u_h) + E_{\Gamma_0}(u_h) + E_{\Gamma_-}(u_h)) \| z \|_{1, \Omega}
\]
\[
\leq C(E_{\Omega}^2(u_h) + E_{\Gamma_0}^2(u_h) + E_{\Gamma_-}^2(u_h))^\frac{1}{2} \| z \|_{1, \Omega}.
\]
(3.17)

Finally, using (3.6) and (3.17) we have
\[
\| u - u_h \|_{1, \Omega} = \sup_{\psi \in H^1_0(\Omega)} \frac{|(u - u_h, \psi)|}{\| \psi \|_{1, \Omega}} \leq C \frac{|(u - u_h, \psi)|}{\| z \|_{1, \Omega}}
\]
\[
\leq C(E_{\Omega}^2(u_h) + E_{\Gamma_0}^2(u_h) + E_{\Gamma_-}^2(u_h))^\frac{1}{2}.
\]
(3.18)

This implies the theorem result.
In Theorem 3.2 we derive a posteriori bound for $u - u_h$ in the negative norm. This norm is usually not computable and it is weaker than $L^2$ norm, $H^1$ norm and above energy, but it is well known that the perturbation nature of finite element approximates can be captured in terms of the negative norm (see [38,39]). So a posteriori error estimator in negative norm can be used to guide the generation of adaptive meshes.

4 Numerical example

In this section, three numerical experiments are carried out to illustrate the above theoretical results. Based on the uniform mesh partition we compute two dimensional transport equations by using discontinuous Galerkin method (DG) and discontinuous streamline diffusion method (DSD). Based on our a posteriori error bound of discontinuous streamline diffusion method, we design and implement the corresponding adaptive algorithm to ensure reliable and efficient control of the error within a given tolerance. In the following examples, We consider two typical errors, $H^1$ norm error and boundary error:

$$\sum_{\tau \in T_h} \|u - u_h\|_{1,\tau}, \quad \sum_{\tau \in T_h} \|\langle u - u_h\rangle a \cdot n\|_{0,\partial\tau}.$$ 

Form the priori error estimate (2.19), we know that the rate of convergence of the above errors are $\min\{r,k\}$ and $\min\{r,k\} + 1/2$ for DG and DSD methods, respectively.

In adaptive discontinuous streamline diffusion method, the convergence rate is measured by the total number of nodes $N$, since the mesh is not uniform. We use a function $F(N) = CN^{-p/2}$ to estimate the order of convergence $p$.

**Example 4.1.** Consider the transport problem (2.1) on the unit square $\Omega = [0,1] \times [0,1]$. Let $a = (\sqrt{2}/2, \sqrt{2}/2)$, $b = 2$. The exact solution $u$ is chosen to be

$$u = x^2y^2$$

and $f$ and $g$ are determined from $u$ so that (2.1) is satisfied.

It is easy to see that the exact solution $u$ given in (4.1) is an analytic function. We shall use the DSD method with $\delta = h$ to solve the example. In Fig. 1, we present the log-log plots of the rate of convergence for DG and DSD methods on uniformly refined mesh. This numerical experiment show that DSD method can improve the accuracy of numerical solution in comparison with the DG method.

**Example 4.2.** Consider the transport problem (2.1) on the unit square $\Omega = [0,1] \times [0,1]$. Let $a = (1/2, \sqrt{3}/2)$, $b = 1$. The exact solution $u$ is chosen to be

$$u = -0.2\sin(2\pi x)\cos(2\pi y) + 0.5\exp(20r^2)\cos\left(\frac{3r^2}{2}\right) - 10,$$

where $r = \sqrt{(x-0.5)^2 + (y-0.5)^2}$. $f$ and $g$ are determined from $u$ so that (2.1) is satisfied.
It is easy to see that the exact solution $u$ given in (4.2) is a analytic function, i.e., certainly, $u \in H^2(\Omega)$. But in Fig. 4(a) we find that this solution exhibits a interior layer and thus has large gradients near the interior layer. We shall use the DSD method with $\delta = 3h$ to solve the example. In Fig. 2, we present the log-log plots of the rate of convergence for the uniformly refined for DG and DSD methods. This numerical experiment show
that DSD method can improve the accuracy of numerical solution in comparison with the DG method. Then we implement a simple adaptive algorithm. We use the a posteriori error indicators \( \eta^2_{\Omega}(u_h) + \eta^2_{\Gamma_0}(u_h) + \eta^2_{\Gamma_-}(u_h) \) in this case. Here we compare the errors on uniform meshes and adaptive meshes. In Fig. 3, we present the log-log plots of the rate of convergence for the uniformly refined and adaptively refined meshes by using DSD method. One can observe that the adaptive method is much more efficient than the uniform refinement strategy. Fig. 4(b) depicts the refined meshes at some levels generated by adaptive algorithm, which shows that our a posteriori error indicator can exactly capture the boundary layer.

**Example 4.3.** Consider the transport problem (2.1) on the unit square \( \Omega = [0,1] \times [0,1] \). Let \( a = (\sqrt{2}/2, \sqrt{2}/2) \), \( b = 0.1 \). The exact solution \( u \) is chosen to be

\[
    u = \exp \left( 0.1 r \arcsin \left( \frac{y+1}{r} \right) \right) \arctan \left( \frac{r-1.5}{\epsilon} \right),
\]

where \( r = \sqrt{x^2 + (y+1)^2} \), \( \epsilon = 3 \times 10^{-4} \). \( f \) and \( g \) are determined from \( u \) so that (2.1) is satisfied.

When \( \epsilon \) is very small, we know that the exact solution (4.3) exhibits a sharp layer. In the case we choose \( \epsilon = 3 \times 10^{-4} \), here the layer is so stiff that it can be considered as a discontinuity. This can be seen in Fig. 7(a). We shall use the DSD method with \( \delta = 5h \) to solve this example. From Fig. 5 we find that the rate of convergence is rather poor by using uniform refinement. We use the a posteriori error indicators \( E^2_\Omega(u_h) + E^2_{\Gamma_0}(u_h) + E^2_{\Gamma_-}(u_h) \) in this case. In Fig. 6, we present the log-log plots of the rate of convergence for the
uniformly refined and adaptively refined meshes by using DSD method. One observes that the adaptive method is much more efficient than the uniform refinement strategy. Fig. 7(b) depicts the refined meshes at some levels generated by adaptive algorithm. Here the strong interior layer is fully resolved under adaptive refinement meshes, and the method is optimally convergent.
Example 4.4. Consider the transport problem (2.1) on the unit square $\Omega = [-1,1] \times [-1,1]$. Let $a = (\sqrt{2}/2, \sqrt{2}/2)$, $b = 1$. The exact solution $u$ is chosen to be

\[
    u = \begin{cases} 
        0.5 \sin(0.25 \pi (x+1)^2) \sin(0.5 \pi (y-x)), & x \leq y, \\
        \exp(-10(x^2 + (y-x)^2)), & x > y.
    \end{cases}
\]  

(4.4)

$f$ and $g$ are determined from $u$ so that (2.1) is satisfied.
We know that the exact solution (4.4) is discontinuous along the line $y = x$. This can be observed in Fig. 10(a). We shall use the DSD method with $\delta = 3h$ to solve this example. We choose triangular meshes that the discontinuous lies on element interfaces. From Fig. 8 we find that the rate of convergence is optimal by using uniform refinement. We use the a posteriori error indicators $E^2_\Omega(u_h) + E^2_{\Gamma_0}(u_h) + E^2_{\Gamma_-}(u_h)$ in this example. In Fig. 9, we present the log-log plots of the rate of convergence for the uniformly refined and adaptively refined meshes by using DSD method. One observes that the adaptive method is much more efficient than the uniform refinement strategy. Fig. 10(b) depicts the refined meshes at some levels generated by adaptive algorithm. But we note that if the mesh is not aligned with discontinuity or the direction $a$ is not parallel to the discontinuity, then the DSD method converges at the same slow rate as the DG method. In [27, 40], they also point out that the convergence rate is limited by the regularity of $u$.

5 Conclusions

We have proposed the residual-based a posteriori error estimates of discontinuous streamline diffusion method for transport equations. We have been able to construct computable upper bounds on the error measured in terms of a natural mesh-dependent energy norm and negative norm. We noted that a posteriori error estimators use directly all the available information from the discontinuous streamline diffusion method, and can be computed efficiently. This direct information includes interior residual errors and boundary residual errors. Numerical examples have demonstrated the effectiveness of our two posteriori error estimators.
Figure 9: Log-log plots of the rate of convergence for uniform meshes and adaptive meshes by DSD in Example 4.4, $H^1$ norm error method (a), Boundary error (b).

Figure 10: The exact solution (a) and an adaptively refined meshes with 41645 nodes (b) by DSD in Example 4.4.

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References


