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A Posteriori Error Estimates of Discontinuous Streamline Diffusion Methods for Transport Equations

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Abstract. Residual-based posteriori error estimates for discontinuous streamline diffusion methods for transport equations are studied in this paper. Computable upper bounds of the errors are measured based on mesh-dependent energy norm and negative norm. The estimates obtained are locally efficient, and thus suitable for adaptive mesh refinement applications. Numerical experiments are provided to illustrate underlying features of the estimators.

AMS subject classifications: 65N30

Key words: A posteriori error estimates, discontinuous streamline diffusion methods, transport equations.

1 Introduction

A posteriori error estimates and corresponding adaptive computation have been an active research field in recent years, especially for elliptic and parabolic equations. Since the pioneering work of Babuška and Rheinboldt [1], a large number of work were devoted to develop a posteriori error estimates and adaptive algorithms. Residual type a posteriori error estimates have been developed, e.g., in [1–8]. Recovery type a posteriori error estimates have been established in [9–13]. For more references about a posteriori error estimate one can refer to [14–18].

As we know the standard Galerkin finite element approximation of transport equation often leads to interior or boundary layer where the gradient of solution changes

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rapidly. Therefore, various stabilization methods have been developed, e.g., discontinuous Galerkin methods [19–21], streamline diffusion methods [22, 23], finite volume methods [24], interior penalty finite element methods [25] and discontinuous streamline diffusion method [26]. Compared with elliptic and parabolic equation a posteriori error estimate for the transport equation is far less developed in the literature. This is mainly due to the fact that the elliptic problem has some smoothing and stability properties. A posteriori error estimate is mainly based on the stability of the problem on the continuous level. The transport equation's stability on the continuous level is not as good as sample elliptic problems. A posteriori error estimates for discontinuous Galerkin methods and streamline diffusion methods of transport equations based on dual argument and nonstandard norm can be found, e.g., in [27–29]. Based on the saturation assumption and interpolation estimate between discrete space a posteriori error estimates for the transport equation have been presented in [25, 30]

In this paper we mainly focus on developing a posteriori error estimate for discontinuous streamline diffusion approximation of transport equations. This method not only keeps the advantage of the upwind approach, but also improves the stability of discontinuous Galerkin method. The stability and a priori error estimate in certain norms of the discontinuous streamline diffusion method can be found in [26, 31]. Residual type a posteriori error estimates in mesh dependent norm and H^{-1} norm for discontinuous streamline diffusion approximation of transport equations are derived. Numerical examples are given to illustrate the theoretical findings.

The paper is organized as follows. In the next section we briefly recall the discontinuous streamline diffusion approximation of transport equation including stability estimate and a priori error estimate. In Section 3 we provide two residual type a posteriori error estimates of discontinuous streamline diffusion methods in mesh dependent norm and H^{-1}-norm. In Section 4 we provide several numerical tests which support our theory. Finally, in Section 5 we summarize the work presented in this paper.

Throughout this paper, some standard notations are used for Sobolev spaces, and the corresponding semi-norms and norms [32]. Moreover, the letter C denotes a generic constant which may stand for different values at its different occurrences and is independent of the mesh parameters.

2 Preliminaries

2.1 The discontinuous streamline diffusion method

Let Ω be a polygon in $\mathbb{R}^2$ with a boundary Γ. Suppose that $\mathbf{a}=(a_1,a_2)$ is a vector function defined on $\overline{\Omega}$ with $a_1,a_2\in W^1_\infty(\Omega)$, and consider the following subsets of Γ:

$$\Gamma_-=\{x\in\Gamma:\mathbf{a}\cdot\mathbf{n}(x)<0\},\quad \Gamma_+=\{x\in\Gamma:\mathbf{a}\cdot\mathbf{n}(x)\geq 0\},$$

where $\mathbf{n}(x)$ is the unit outward normal at the point $x\in\partial\Omega$. The sets Γ_- and Γ_+ are referred as the inflow and outflow boundary, respectively. We consider the following transport