Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2017-0183

Vol. **24**, No. 5, pp. 1375-1408
November 2018

A Positivity-Preserving Finite Volume Scheme for Heat Conduction Equation on Generalized Polyhedral Meshes

Hui Xie[1], Xuejun Xu[2,3], Chuanlei Zhai[1] and Heng Yong[1,*]

[1] *Institute of Applied Physics and Computational Mathematics, Beijing, 100094, China.*
[2] *LSEC, Institute of Computational Mathematics, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, P.O. Box 2719, Beijing, 100190, China.*
[3] *School of Mathematical Sciences, Tongji University, Shanghai, China.*

Received 30 August 2017; Accepted (in revised version) 11 December 2017

Abstract. In this paper we present a nonlinear finite volume scheme preserving positivity for heat conduction equations. The scheme uses both cell-centered and cell-vertex unknowns. The cell-vertex unknowns are treated as auxiliary ones and are eliminated by our newly developed second-order explicit interpolation formula on generalized polyhedral meshes. With the help of the additional parameters, it is not necessary to choose the stencil adaptively to obtain the convex decomposition of the co-normal vector and also is not required to replace the interpolation formula with positivity-preserving but usually low-order accurate ones whenever negative interpolated auxiliary unknowns appear. Moreover, the new flux approximation has a fixed stencil. These features make our scheme more efficient compared with other existing methods based on Le Potier's nonlinear two-point approximation, especially in 3D. Numerical experiments show that the scheme maintains the positivity of the continuous solution and has nearly second-order accuracy for the solution on the distorted meshes where the diffusion tensor may be anisotropic and discontinuous.

AMS subject classifications: 65M08, 35R05, 76S05

Key words: Heat conduction, anisotropic diffusion tensor, distorted mesh, positivity-preserving, cell-centered scheme, vertex elimination.

1 Introduction

In the simulation of Lagrangian radiation hydrodynamics [20], both the accuracy and positivity-preserving property are the key factors to any discrete scheme for heat con-

*Corresponding author. *Email addresses:* xiehui@lsec.cc.ac.cn (H. Xie), xxj@lsec.cc.ac.cn (X. Xu), zhai_chuanlei@iapcm.ac.cn (C. L. Zhai), yong_heng@iapcm.ac.cn (H. Yong)

duction equations. Due to natural local conservation and easy coupling with the cell-centered hydrodynamics computation, finite volume methods [13] are very popular ones among the various numerical methods. Many practical problems we confront [8, 20] are multiscale in company with the large deformation of the fluid flow, these phenomena challenge the robustness and accuracy of the numerical methods. There are many literatures [1, 19, 27, 36] devoted to design the linear finite volume method to recover the optimal convergence rate on the distorted meshes either with or without material discontinuities. A comprehensive review of these improved finite volume methods is present in [12].

As for positivity-preserving property, it is another key requirement for the discretization schemes. In the context of heat conduction, the scheme without preserving positivity may render the negative temperature, then related physical variables, such as density and material temperature, become negative which will terminate the whole simulation. For the linear schemes, the positivity-preserving property is equivalent with the discrete maximum principle. It is well known that the classical linear finite volume (FV) and finite element (FE) scheme may fail to satisfy the discrete maximum principle [11, 30]. Some research efforts have been made to achieve this property.

In the context of the FE method, the first work concerning this property can be tracked to the seminal paper by Ciarlet and Raviart [9], which gave the geometry conditions to meet the maximum principle for *isotropic* diffusion equation. A comprehensive review of recent works related to the mesh restriction conditions can be found in [29]. To avoid altering the mesh for different diffusion tensors, some new weak formulations were developed to meet the maximum principle. A stabilized weak formulation [17] was constructed for *isotropic* parabolic problems and extended to the transient advection-diffusion-reaction problems [18]. A *space-time* discontinuous Galerkin method [35] was also developed to meet non-negativity constraint for hyperbolic advection-diffusion problems. Recently, some optimization-based methodologies based on quadratic programming [3, 26, 31] have been shown to satisfy the maximum principle and the non-negative constraint for the anisotropic diffusion problem without changing the weak formulation and been extended to advection-diffusion-reaction equations [28] and transient diffusion equations [32]. Furthermore, to handle more general transport problem (e.g., non-self-adjoint) and weak formulations (e.g., non-symmetric), weak formulations have been rewritten as a variational inequality with the feasible solution space restricted by the maximum principle [5]. In the context of the FV scheme, the restriction conditions for mesh and diffusion tensor to meet the non-negativity constraint were analyzed in [30] for the classical linear FV scheme. A nonlinear correction for FV schemes [4] was proposed to enforce the discrete maximum principle. However, it was shown in [19, 30] that no linear nine-point scheme can unconditionally satisfy the positivity-preserving property with the second-order accuracy on any distorted quadrilateral mesh or for any diffusion tensor. A nonlinear discretization scheme might be a price to pay for the construction of positivity-preserving finite volume scheme with the second-order accuracy.

In the last decade, some nonlinear positivity-preserving finite volume methods have