An Alternating Direction Method of Multipliers for the Optimization Problem Constrained with a Stationary Maxwell System

Yongle Hao¹, Haiming Song¹, Xiaoshen Wang² and Kai Zhang¹,*

¹ Department of Mathematics, Jilin University, Changchun 130012, P.R. China.
² Department of Mathematics and Statistics, University of Arkansas at Little Rock, Little Rock, Arkansas 72204, United States.

Received 23 May 2017; Accepted (in revised version) 13 November 2017

Abstract. This paper mainly focuses on an efficient numerical method for the optimization problem constrained with a stationary Maxwell system. Following the idea of [32], the edge element is applied to approximate the state variable and the control variable, then the continuous optimal control problem is discretized into a finite dimensional one. The novelty of this paper is the approach for solving the discretized system. Based on the separable structure, an alternating direction method of multipliers (ADMM) is proposed. Furthermore, the global convergence analysis is established in the form of the objective function error, which includes the discretization error by the edge element and the iterative error by ADMM. Finally, numerical simulations are presented to demonstrate the efficiency of the proposed algorithm.

AMS subject classifications: 90C30, 90C33, 65K10, 65M60

Key words: Optimal control problem, stationary Maxwell’s equations, Nédélec element, ADMM.

1 Introduction

Many complicated problems in engineering, mathematical finance, physics, and life sciences could be modeled by optimization problems with partial differential equations (PDEs) as constraints. The rising number of real world applications demands further developments in numerical schemes for PDE-constrained optimization problems. Thus far, the numerical method based on finite difference methods, finite volume methods, etc., have been developed for the elliptic equation or parabolic equation constrained optimizations and we refer [6,9–12,37], and references therein for the rich literature.

*Corresponding author. Email addresses: kzhang@jlu.edu.cn (K. Zhang), haoyl1009@mails.jlu.edu.cn (Y. Hao), songhaiming@jlu.edu.cn (H. Song), xxwang@ualr.edu (X. Wang)
The stationary Maxwell equations play important roles in many modern technologies and applications such as fusion energy, magnetohydrodynamics, electromagnetic induction heating, signal processing, magnetic levitation, and so on. Since the pioneering works of mathematicians and engineers, including Nédélec [22, 23] and Monk [20], the edge element method for solving Maxwell’s equations has been widely accepted in the field of mathematical sciences as well as engineering. And for other numerical methods, please see [1, 16, 21, 25, 26, 31, 36] for the details. Recently, the optimal control of Maxwell’s equations has attracted researchers’ attentions. For the complexity of this kind of problems, the existing researches are mainly based on the degenerate Maxwell’s equations [30, 32, 33]. Here, we also focus on the optimization problem constrained with a stationary Maxwell system, and propose an efficient numerical algorithm with the help of the analysis on the edge element in [32] and ADMM in [5].

Generally speaking, there are two strategies for solving the stationary Maxwell constrained optimization problem as well as the traditional PDE-constrained optimization problems: discretize-then-optimize and optimize-then-discretize algorithms [6]. It is well known that the two strategies are equivalent in the sense that the discretised system of the continuous optimality conditions coincides with the optimality conditions for the discretised minimization problem [33], but they have differences in terms of system structure. The latter approach is mainly based on solving a continuous variational system and the aim of the former is to deal with a discretised optimization problem, which could be solved by other optimization algorithms except for the first order optimal condition. Furthermore, based on whether the state variable can be expressed as a function of the control variable directly or not, the numerical methods are divided into reduced space methods and full space methods. Full space methods often are used to solve the optimal control problems with stationary or nonlinear PDE constraints, and reduced space methods are mainly applied to solve the problems with memory requirements or time-dependent problems.

Here, we only mention two representative algorithms for the stationary Maxwell constrained optimization problems. One is the standard gradient descent method, which is a reduced space method with the discretize-then-optimize strategy. The existence, uniqueness, and regularity of the optimal solution for the Maxwell constrained optimization problems with the pointwise state constraints are shown in [32]. Yousept adopted the edge element method to discretize the reduced problem, and solve the discretized system by the gradient descent method. He also presented the error estimates and obtained a reasonable numerical performance. The other one is the adaptive edge element method, which is a full space method with the optimize-then-discretize strategy. Xu and Zou proposed an adaptive edge element method to solve the KKT system associated with the original optimal control problem [30]. A posteriori error estimator of the residue type is derived for the lowest-order edge element approximation. They have also proved that the sequence of discrete solutions converges strongly to the exact solution and the error estimator has a vanishing limit. We refer to [2–4, 24] for more details on the theoretical and computational analysis for the optimization problem constrained by Maxwell’s
There exist two main challenges for the numerical treatment of the optimal control problem with stationary Maxwell’s equation constraint:

(i) The resulting discretized problem is a complex coupled system, which consists of state variable, control variable, and Lagrangian multiplier. If the full space method is used, it requires dealing with a large-scale system, which can not be implemented easily and efficiently. If the reduced space method is applied, the derivative of the state variable with respect to the control variable needs to be determined by the dual equation, which will increase the cost of computations. There are some efficient algorithms for this problem, such as the semismooth Newton method [4], the quasi Newton method [28] and the SQP method [34] etc. We will propose an alternative method—ADMM.

(ii) The convergence analysis is the theoretical guarantee for the performance of the algorithm. For the complexity of the problem, some iterative methods are needed. However, up to now, there is no global error estimates which consists of the discretized error of edge element method and the iteration error for solving the discretized system. For the traditional ADMM analysis, the error estimate between two consecutive iterative solutions (either in the ergodic sense or the nonergodic sense) have been presented, but it is not the suitable formula for the global error analysis.

Based on the discretize-then-optimize strategy, we propose a highly efficient and stable numerical algorithm to resolve the two issues above. For the first challenge, we discretize the continuous optimal control problem by the edge element method, then reformulate it in the matrix-vector form. We emphasize that the discretized problem is essentially a convex optimization problem with separable structure, where the alternating direction method of multipliers (ADMM) is an efficient approach for this kind of problems [12]. The ADMM divides the original large-scale coupled system into several smaller subproblems, which could be solved easily. For its high efficiency, this algorithm has been applied to many engineering and industrial fields, such as image restoration and denoising, sparse signal recovery, machine learning [8], regularized estimation [29], compressed sensing [38], etc.. The convergence analysis of the ADMM has also been studied systematically, we refer to [5, 14, 15] for more details. In this paper, we shall use the ADMM to solve the discretized system of the optimal control problem. For the second challenge, we present convergence analysis with respect to the objective function error, and this kind of error estimates implies the global error estimates directly. Firstly, the edge element method discretization error is $O(h^{\frac{1}{2}+\epsilon})$ by the standard technique [17,21], where $h$ is the partition size in spacial direction and $\epsilon \in (0,0.5]$. Then, based on the idea in [5], the iterative error of the ADMM is $O(\frac{1}{k})$, where $k$ is the number of iteration steps. Finally, the global error estimate is established from the triangle inequality. We point out that the numerical simulations of the ADMM always show much better
convergence rate than the theoretical estimate. In this paper, the ADMM is applied to solve the optimization problem constrained with stationary Maxwell’s equations, which avoids solving large-scale coupled system and improves the efficiency. To the best of our knowledge, there are few works on the global error estimate with respect to the objective function. Our error estimate technique can be generalized to the other PDE-constrained optimizations.

The rest of this paper is organized as follows. In Section 2, we shall describe the optimal control problem with stationary Maxwell’s equation as the constraint, and show the regularity property of the solution for this model problem. In Section 3, the edge element method is applied to approximate the model problem. The discretized error analysis is also presented. The ADMM for the discretized system and the corresponding iterative error are given in Section 4. Moreover, the global convergence analysis is also established. In Section 5, Numerical simulations are presented to test the performance of the proposed method. The last section is devoted to some concluding remarks.

2 The model problem

For the convenience of the expression, we introduce some definitions and notations firstly. Let \( \Omega \in \mathbb{R}^3 \) be a bounded, simply connected Lipschitz polyhedral domain with a connected boundary \( \partial \Omega \). Denote by \( V \) the Hilbert space of three-dimensional vector functions. Let \( \| \cdot \|_V \) and \( (\cdot, \cdot)_V \) be the associated norm and the associated inner product of \( V \). Unless stated otherwise, \( \| \cdot \| \) and \( (\cdot, \cdot) \) stands for \( \| \cdot \|_{L^2} \) and \( (\cdot, \cdot)_{L^2} \) respectively. The following spaces are used throughout the paper:

\[
\begin{align*}
C^{0,1}(\overline{\Omega}) &= \{ f \in C(\overline{\Omega}) \mid \exists \ C > 0, \ \forall \ x, y \in \Omega, \ |f(x) - f(y)| \leq C \|x - y\| \}, \\
H(\text{curl}) &= \{ z \in L^2(\Omega) \mid \text{curl}z \in L^2(\Omega) \}, \\
H_0(\text{curl}) &= \{ z \in H(\text{curl}) \mid z \times n = 0 \text{ on } \partial \Omega \}, \\
H^s(\text{curl}) &= \{ z \in H^s(\Omega) \mid \text{curl}z \in H^s(\Omega) \}, \quad s > 0, \\
H_0^s(\text{curl}) &= H^s(\text{curl}) \cap H_0(\text{curl}), \quad s > 0.
\end{align*}
\]

Here, the curl-operator and the tangential trace are understood in the sense of distributions [21].

With the above notions, the optimization problem constrained by stationary Maxwell’s equations states as follows:

\[
\begin{align*}
\min_{(y,u) \in H_0(\text{curl}) \times H_0(\text{curl})} \mathcal{J}(y,u) &= \frac{1}{2} \|y - y_d\|^2 + \frac{\tau}{2} \|\text{curl}y - y_c\|^2 + \frac{\kappa}{2} \|u\|^2_{H(\text{curl})} \\
\text{subject to} \quad &\begin{cases} \text{curl}(\alpha \text{curl}y) + \beta y = \text{curl}u, & \text{in } \Omega, \\
n \times y = 0, & \text{on } \partial \Omega, \end{cases} \\
\text{with the pointwise state constraints} \quad &y_a(x) \leq y(x) \leq y_b(x) \quad \text{a.e. in } \Omega.
\end{align*}
\]
Here, $y_\alpha, y_\beta, y_c, y_d \in L^2(\Omega)$, $\tau$ and $\kappa$ are constants, and the coefficients $\alpha, \beta \in L^\infty(\Omega)$ satisfy
\[
\alpha_1 \leq \alpha(x) \leq \alpha_2 \quad \text{and} \quad \beta_1 \leq \beta(x) \leq \beta_2 \quad \text{a.e. in } \Omega,
\]
with some constants $0 < \alpha_1 \leq \alpha_2 < +\infty$ and $0 < \beta_1 \leq \beta_2 < +\infty$. The optimal control problem (2.1)-(2.2) has extensive applications in earth physics, life science, remote sensing technology, nondestructive testing, material science, and other engineering, medical and military fields.

Notice that the constraint equation (2.2) is the Maxwell's equation with the implicit time-stepping [25]. By the traditional analysis, there exists a well-defined linear and bounded operator $S: H_0^0(\text{curl}) \rightarrow H_0^0(\text{curl})$ associated with (2.2) such that $y = Su$. Under a Lipschitz-continuity assumption on the coefficients $\alpha$ and $\beta$, a higher regularity property for the solution operator can be obtained, and summarized as follows:

**Lemma 2.1** ([32]). (i) Let $\beta \in C_0^1(\overline{\Omega})$, then there exists a positive $\epsilon$, such that the solution operator $S$ from $H_0^0(\text{curl})$ to $H_0^0(\text{curl}) \cap H^{1+\epsilon}(\text{curl})$ is well-defined and bounded.

(ii) Let $\alpha, \beta \in C_0^1(\overline{\Omega})$, then there exists a positive $\epsilon$, such that the solution operator $S$ from $H_0^0(\text{curl})$ to $H_0^{1+\epsilon}(\text{curl})$ is well-defined and bounded. Further, if $\Omega$ is convex, then $\epsilon = \frac{1}{2}$.

With the solution operator $S$, the optimal control problem (2.1)-(2.3) can be rewritten as follows:
\[
\min_{u \in U} f(u) = \frac{1}{2} \|Su - y_d\|^2 + \frac{\tau}{2} \|\text{curl}Su - y_c\|^2 + \frac{\kappa}{2} \|u\|^2_{H(\text{curl})}, \tag{2.4}
\]
where the feasible set $U$ is given by the following closed convex set:
\[
U := \{u \in H_0^0(\text{curl}) \mid y_a(x) \leq Su(x) \leq y_b(x) \text{ a.e. in } \Omega\}.
\]
Here, we assume that $U$ is not empty. For instance, if $y_a(x) \leq 0$ and $y_b(x) \geq 0$ hold a.e. in $\Omega$, then $0 \in U$.

The optimization problem (2.4) is a linear quadratic optimal control problem, the existence and uniqueness results for this kind of problem can be found in [17, 19, 27]. Moreover, the solution of the optimal problem (2.4) satisfies the following regularity property:

**Lemma 2.2** ([32]). Let $u^*$ be the optimal solution of (2.4), then we have
\[
u^* \in H_0^0(\text{curl}) \cap H^{1+\epsilon}(\Omega),
\]
where $\epsilon > 0$ is given in Lemma 2.1.

Following [18], we use the Moreau-Yosida regularization to deal with the pointwise state constraints (2.3). The corresponding regularized optimal control problem related to (2.1)-(2.3) reads as follows:
\[
\min_{u \in U} f_\gamma(u) = f(u) + \frac{\gamma}{2} (\|\max(y_a - Su, 0)\|^2 + \|\max(0, Su - y_b)\|^2), \tag{2.5}
\]
where $\gamma > 0$. The regularity of the optimal solution $(u_\gamma)_{\gamma > 0}$ of problem (2.5) is given by:
Lemma 2.3 ([32]). Let $\varepsilon > 0$ be given in Lemma 2.1. Then $u_\gamma \in H^{1+\varepsilon}_0(\text{curl})$ holds for all $\gamma > 0$.

The strong convergence of $(u_\gamma)_{\gamma > 0}$ towards $u^*$ under $H(\text{curl})$-norm as $\gamma \to +\infty$ has been established in [32].

3 The discretized problem

Here, we recall the original optimization problem (2.1)-(2.3) for the discretized problem. Firstly, we give an equivalent form of the problem (2.5), and the corresponding variational formulation is presented. Then, we adopt the edge element to discretize the variational formulation, and the corresponding error estimates with respect to the objective function are also established.

3.1 Edge element discretization

Using the solution operator $S$, the Moreau-Yosida regularized optimal control problem corresponding to (2.5) is given by

$$
\min_{(y,u) \in H_0(\text{curl}) \times H_0(\text{curl})} \mathcal{J}_\gamma(y,u),
$$

subject to $(\alpha \text{curl} y, \text{curl} z) + (\beta y, z) = (\text{curl} u, z), \quad \forall z \in H_0(\text{curl}),
$$

where

$$
\mathcal{J}_\gamma (y,u) = \mathcal{J}(y,u) + \frac{\gamma}{2} \left( \| \max(y_a - y_h,0) \|^2 + \| \max(0,y - y_h) \|^2 \right).
$$

Now, we introduce the edge element approximation for the variational system (3.1)-(3.2). Let $T_h$ be a regular triangulation consisting of tetrahedron $T$ such that $\Omega = \bigcup_{T \in T_h} T$. Here $h$ stands for the maximal diameter of all elements $T$. We define the space of the lowest order edge element by

$$
V_h := \{ z_h \in H_0(\text{curl}) \mid z_{hT} = a + b \times x \text{ with } a, b \in \mathbb{R}^3 \}.
$$

Let $\Phi_i, i = 1, \cdots, N$ be the basis functions of the lowest order edge element space $V_h$, then the discretized control variable and state variable could be expressed as

$$
u_h = \sum_{i=1}^N u_i \Phi_i, \quad y_h = \sum_{i=1}^N y_i \Phi_i,
$$

where $y_i$ and $u_i, i = 1, \cdots, N$ are the undetermined coefficients. The variational system (3.1)-(3.2) can be approximated by: Find $y_h, u_h \in V_h$ such that

$$
\min_{(y_h,u_h) \in V_h \times V_h} \mathcal{J}_\gamma (y_h,u_h) = \frac{1}{2} \| y_h - y_d \|^2 + \frac{\gamma}{2} \| \text{curl} y_h - y_c \|^2
$$

$$
+ \frac{\alpha}{2} \| u_h \|^2_{H(\text{curl})} + \frac{\gamma}{2} \left( \| \max(y_a - y_h,0) \|^2 + \| \max(0,y - y_h) \|^2 \right),
$$

$$(\alpha \text{curl} y_h, \text{curl} z_h) + (\beta y_h, z_h) = (\text{curl} u_h, z_h), \quad \forall z_h \in V_h.
$$

(3.4)
This is essentially a finite dimensional optimization problem.

3.2 The error estimate

Let \((u_\gamma, y_\gamma)\) and \((u_{\gamma h}, y_{\gamma h})\) stand for the optimal solutions of the systems (3.1)-(3.2) and (3.4)-(3.5) respectively. \(S_h : H_0(\text{curl}) \to V_h\) denote the solution operator associated with the discrete state equation. More precisely, for any \(u \in H_0(\text{curl})\), \(y_h = S_h u \in V_h\) is the solution of

\[
(a \nabla \times \vec{y}_h, \nabla \times \vec{z}_h) + (\beta y_h, \vec{z}_h) = (\nabla \times u, \vec{z}_h), \quad \forall \vec{z}_h \in V_h. \quad (3.6)
\]

Then, we have the following error estimates:

**Lemma 3.1 ([32]).** Let \(\gamma > 0\) and \(\alpha, \beta \in C^{0,1}(\Omega)\). Then there exist constants \(C_1, C_2 > 0\) independent of \(h\) such that

\[
\|u_\gamma - u_{\gamma h}\|_{H(\text{curl})} \leq C_1 h^{1+\varepsilon} \|u_\gamma\|_{H^{1+\varepsilon}(\text{curl})}, \quad (3.7)
\]

\[
\|S u - S_h u\|_{H(\text{curl})} \leq C_2 h^{1+\varepsilon} \|u\|_{H(\text{curl})}, \quad \forall u \in H_0(\text{curl}), \quad (3.8)
\]

where \(\varepsilon > 0\) is given in Lemma 2.1. In particular, if \(\Omega\) is convex, then we obtain

\[
\|u_\gamma - u_{\gamma h}\|_{H(\text{curl})} \leq C_1 h \|u_\gamma\|_{H^{1}(\text{curl})}. \quad (3.9)
\]

Lemma 3.1 presents the error between the continuous control \(u_\gamma\) and its discretized form \(u_{\gamma h}\). Further, as a standard consequence, the edge element error on the state variable and the objective function can be presented as follows:

**Corollary 3.1.** Let \(\gamma > 0\), \(\alpha, \beta \in C^{0,1}(\Omega)\), and \(h < 1\). Then there exist constants \(C_3, C_4 > 0\) independent of \(h\) such that

\[
\|y_\gamma - y_{\gamma h}\|_{H(\text{curl})} \leq C_3 h^{1+\varepsilon} \|u_\gamma\|_{H^{1+\varepsilon}(\text{curl})}, \quad (3.10)
\]

\[
|J_\gamma(y_\gamma, u_\gamma) - J_\gamma(y_{\gamma h}, u_{\gamma h})| \leq C_4 h^{1+\varepsilon}, \quad (3.11)
\]

where \(\varepsilon > 0\) is given in Lemma 2.1 and

\[
C_4 = \left[ \frac{1}{2} (1 + \tau + 2\gamma) (C_3 \|u_\gamma\|_{H^{1+\varepsilon}(\text{curl})} + 2\|y_\gamma\|_{H(\text{curl})}) \right. \\
+ \left( (\|y_d\| + \tau) \|y_c\| + \gamma \|y_a\| + \gamma \|y_b\|) \right. \\
+ \left. C_1 \kappa \left( C_1 \|u_\gamma\|_{H^{1+\varepsilon}(\text{curl})} + 2\|u_d\|_{H(\text{curl})} \right) \right] \|u_\gamma\|_{H^{1+\varepsilon}(\text{curl})}. \quad (3.12)
\]

The proof of Corollary 3.1 are provided for the completeness in the appendix [32]. Now, we are at the stage to present the convergence analysis for the objective function error between the system (3.1)-(3.2) and their discretized system (3.4)-(3.5).
4 The ADMM for the discretized system

In this section, we introduce the alternating direction method of multipliers (ADMM) for solving the finite dimensional problem (3.4)-(3.5), and present the convergence analysis for the proposed method.

4.1 The simplified optimization problem

For the convenience, we present the vector form corresponding to the finite dimensional optimization problem (3.4)-(3.5) firstly. Recall the edge element proximation (3.3), and let
\[ R = (\Phi_1, \cdots, \Phi_N), \quad u_h = (u_1, \cdots, u_N)^T, \quad y_h = (y_1, \cdots, y_N)^T, \]
then we have
\[ u_h = Ru_h, \quad y_h = Ry_h. \] (4.1)

Then the optimal system (3.4)-(3.5) can be simplified as
\[
\begin{align*}
\min_{(y_h, u_h) \in \mathbb{R}^N \times \mathbb{R}^N} F(y_h, u_h), \\
\text{subject to} \quad Au_h - By_h = 0,
\end{align*}
\] (4.2) (4.3)

where
\[
\begin{align*}
F(y_h, u_h) &= \frac{1}{2} \| Ry_h - y_d \|_2^2 + \frac{T}{2} \| \text{curl} (Ry_h) - y_c \|_2^2 + \frac{\kappa}{2} \| Ru_h \|_{H(\text{curl})}^2 \\
&\quad + \frac{\gamma}{2} (\| \max(y_a - Ry_h, 0) \|_2^2 + \| \max(0, Ry_h - y_b) \|_2^2).
\end{align*}
\] (4.4)

We emphasize that the vector form optimization problem (4.2)-(4.3) has a separable structure. The objective function (4.2) could be reformulated as
\[
\begin{align*}
F(y_h, u_h) &= \theta_1(u_h) + \theta_2(y_h),
\end{align*}
\] (4.5) (4.6)

where
\[
\begin{align*}
\theta_1(u_h) &= \frac{1}{2} \| Ru_h \|_{H(\text{curl})}^2, \\
\theta_2(y_h) &= \frac{1}{2} \| Ry_h - y_d \|_2^2 + \frac{T}{2} \| \text{curl} (Ry_h) - y_c \|_2^2 \\
&\quad + \frac{\gamma}{2} (\| \max(y_a - Ry_h, 0) \|_2^2 + \| \max(0, Ry_h - y_b) \|_2^2).
\end{align*}
\]

Next, we present the convexity of the objective function (4.4), which will be a useful tool subsequently.

**Lemma 4.1.** \( \theta_1 \) and \( \theta_2 \) are convex functions with respect to \( u_h \) and \( y_h \) respectively.

The proof of lemma 4.1 follows from that \( \max(a, 0) = \frac{a + |a|}{2} \) for any real number \( a \).

Furthermore, the solution set of (4.3) is nonempty due to the existence and uniqueness of the operator \( S \). Therefore, the problem (4.2)-(4.3) is a separable convex optimization with linear constraint, which could be solved by ADMM efficiently.
4.2 The convergence analysis of ADMM

The augmented Lagrangian function for the optimization problem (4.2)-(4.3) is given by

\[ L_\rho(u_h, y_h, \lambda) = \theta_1(u_h) + \theta_2(y_h) - \lambda^T (Au_h - By_h) + \frac{\rho}{2} \|Au_h - By_h\|^2, \quad (4.7) \]

where \( \theta_1 \) and \( \theta_2 \) are defined by (4.5)-(4.6).

Then the ADMM for solving (4.2)-(4.3) is given by the following algorithm:

**Algorithm 4.1: ADMM**

Set the initial values \( y_0^h, \lambda_0 \);

**For** \( k = 1, 2, \ldots, K \) **do**

\[ u_{h}^{k+1} = \arg \min_{u_h \in \mathbb{R}^N} L_\rho(u_h, y_h^k, \lambda^k); \]

\[ y_{h}^{k+1} = \arg \min_{y_h \in \mathbb{R}^N} L_\rho(u_{h}^{k+1}, y_h, \lambda^k); \]

\[ \lambda^{k+1} = \lambda^k - \rho (Au_{h}^{k+1} - By_{h}^{k+1}); \]

**End**

Here, \( \lambda \in \mathbb{R}^N \) is the multiplier associated with the linear constraint (4.3) and \( \rho > 0 \) is a penalty parameter. Furthermore, let

\[ \tilde{u}_h^k = \frac{1}{K} \sum_{i=1}^{K} u_i^k, \quad \tilde{y}_h^k = \frac{1}{K} \sum_{i=1}^{K} y_i^k, \quad \tilde{\lambda}^k = \frac{1}{K} \sum_{i=1}^{K} \lambda_i^k, \quad (4.8) \]

then the upper bound of iteration \( K \) in Algorithm 4.1 is determined by the following lemma:

**Lemma 4.2 ([5])**. Let the function \( \theta_1(u_h) \) and \( \theta_2(y_h) \) be convex, and the matrices \( A \) and \( B \) have full column ranks. If the saddle point set \( \mathcal{S} \) of the augmented Lagrangian function \( L_\rho(y_h, u_h, \lambda) \) is nonempty, then the ergodic sequence \( (\tilde{u}_{h}^k, \tilde{y}_{h}^k, \tilde{\lambda}^k) \) generated by Algorithm 4.1 converges to a saddle-point \( (u^*_h, y^*_h, \lambda^*_h) \) in \( \mathcal{S} \), and satisfies

\[ \left| \theta_1(\tilde{u}_h^k) + \theta_2(\tilde{y}_h^k) - [\theta_1(u^*_h) + \theta_2(y^*_h)] \right| \leq \frac{1}{k \min\{2, \rho\}} \max \left\{ \left\| \frac{y^0_h - y^*_h}{\lambda^0 - \lambda^*_h} \right\|_{H^0}, \left\| \frac{y^0_h - y^*_h}{\lambda^0 - \lambda^*_h} \right\|_{\tilde{H}_\rho}, \frac{1}{\rho} \left\| \lambda^0 \right\| \right\}, \]

where

\[ \left\| X \right\|_{\tilde{H}_\rho} = X^T H_\rho X, \quad H_\rho = \begin{bmatrix} \rho B^T B & 0 & 0 \\ 0 & \frac{1}{\rho} I_N \end{bmatrix}, \quad \tilde{H}_\rho = \begin{bmatrix} \frac{1}{\rho} I_N & 0 \\ 0 & \rho B^T B \end{bmatrix}. \]
Remark 4.1. Without the assumption of full column rank of $A$ and $B$, the boundedness of $\{A\tilde{u}_k\}$, $\{B\tilde{y}_k\}$ and $\{\tilde{\lambda}_k\}$ can be derived. The conclusion of Lemma 4.2 still holds if the assumptions can guarantee the boundedness of the generated sequence (cf. [5]).

We claim the following two facts: (i) $(u_{\gamma h}, y_{\gamma h})$ is the unique optimal solution of the optimal control problem (3.4)-(3.5), which implies $(u_{\gamma h}, y_{\gamma h})$ is the unique optimal solution of (4.2)-(4.3), that is

$$J_\gamma(y_{\gamma h}, u_{\gamma h}) = F(y_{\gamma h}, u_{\gamma h}) = \theta_1(u_{\gamma h}) + \theta_2(y_{\gamma h}).$$  

(4.9)

(ii) The ergodic sequence (4.8) generated by Algorithm 4.1 converges to the solution of (4.2)-(4.3), which has the unique optimal solution $(u_{\gamma h}, y_{\gamma h})$. Hence, we have

$$\left(\frac{u_{\gamma h}^*}{y_{\gamma h}^*}\right) = (u_{\gamma h}, y_{\gamma h}).$$  

(4.10)

Based on the two facts above and Lemma 4.2, we can derive

$$|F(\tilde{y}^k_h, \tilde{u}^k_h) - F(y_{\gamma h}, u_{\gamma h})| \leq C_5 \frac{1}{k},$$  

(4.11)

where

$$C_5 = \frac{1}{\min\{2, \rho\}} \max \left\{ \left\| \frac{y^0_h - y^*_{\gamma h}}{\lambda^0 - \lambda^*} \right\|_{H^p}, \left\| \frac{y^0_h - y^*_{\gamma h}}{\lambda^0 - \lambda^*} \right\|_{\tilde{H}^p}, + \frac{1}{\rho} \left\| \lambda^0 \right\| \right\}.$$  

(4.12)

Theorem 4.1. Under the assumptions in Corollary 3.1 and Lemma 4.2, we have the following error estimate

$$|J_\gamma(y_{\gamma}, u_{\gamma}) - F(\tilde{y}^k_h, \tilde{u}^k_h)| \leq C_4 h^{1+\varepsilon} + C_5 \frac{1}{k},$$  

(4.13)

where $C_4$ is given by (3.12) and $C_5$ is defined by (4.12).

Proof. By the triangle inequality, identity (4.9), Theorem 3.1, and the estimate (4.11), we obtain

$$|J_\gamma(y_{\gamma}, u_{\gamma}) - F(\tilde{y}^k_h, \tilde{u}^k_h)|$$

$$\leq |J_\gamma(y_{\gamma}, u_{\gamma}) - J_\gamma(y_{\gamma h}, u_{\gamma h})| + |J_\gamma(y_{\gamma h}, u_{\gamma h}) - F(\tilde{y}^k_h, \tilde{u}^k_h)|$$

$$= |J_\gamma(y_{\gamma}, u_{\gamma}) - J_\gamma(y_{\gamma h}, u_{\gamma h})| + |F(y_{\gamma h}, u_{\gamma h}) - F(\tilde{y}^k_h, \tilde{u}^k_h)|$$

$$\leq C_4 h^{1+\varepsilon} + C_5 \frac{1}{k},$$

which completes the proof.  

4.3 The explicit solution for the iterative sequence of ADMM

In this subsection, we present an explicit iterative sequence related to Algorithm 4.1. The iterative schemes in Algorithm 4.1 can be simplified as

\[
\begin{cases}
\hat{u}_b^{k+1} = \arg \min_{u_b \in \mathbb{R}^N} \theta_1(u_b) + \frac{\rho}{2} \|Au_b - By_b - \hat{\lambda}^k\|_F^2; \\
\hat{y}_b^{k+1} = \arg \min_{y_b \in \mathbb{R}^N} \theta_2(y_b) + \frac{\rho}{2} \|Au_b^{k+1} - By_b - \hat{\lambda}^k\|_F^2; \\
\hat{\lambda}^{k+1} = \hat{\lambda}^k - \rho(Au_b^{k+1} - By_b^{k+1}).
\end{cases}
\]  

(4.14)

Let $y_a$, $y_b$, $y_c$, and $y_d$ stand for the edge element coefficients corresponding to $y_a$, $y_b$, $y_c$, and $y_d$ respectively. The matrices $M_1$, $M_2$, and $M_3$ are defined by

\[
M_1 = \begin{pmatrix} (\Phi_j, \Phi_j) \end{pmatrix} \in \mathbb{R}^{N \times N}, \\
M_2 = \begin{pmatrix} (\text{curl}\Phi_j, \Phi_j) \end{pmatrix} \in \mathbb{R}^{N \times N}, \\
M_3 = \begin{pmatrix} (\text{curl}\Phi_j, \Phi_j) \end{pmatrix} \in \mathbb{R}^{N \times N}.
\]

The ergodic form corresponding to (4.14) can be approximated by using the following explicit iteration algorithm with some simple calculations.

Algorithm 4.2: Explicit iteration for ADMM

Set the initial values $\hat{u}_b^0, \hat{y}_b^0, \hat{\lambda}_b^0$;  

For $k = 1, 2, \cdots, K$ do: 

- $\hat{u}_b^{k+1} = \left(\kappa(M_1 + M_2) + \rho A^T A\right)^{-1}(\rho A^T B y_b^k + A^T \hat{\lambda}^k)$; 
- $\hat{z}_b^{k+1} = (M_1 + \tau M_2 + \rho B^T B)^{-1}(\rho B^T A u_b^{k+1} - B^T \hat{\lambda}^k + M_1 y_d + \tau M_3 y_c)$; 
- $\hat{y}_b^{k+1} = \min(\max(y_d, \hat{z}_b^{k+1}), y_b)$; 
- $\hat{\lambda}^{k+1} = \hat{\lambda}^k - \rho(Au_b^{k+1} - By_b^{k+1})$; 

Obtain $(\hat{u}_b^k, \hat{y}_b^k)$ by the definition (4.8).

Here, we emphasize several facts: (1) Algorithm 4.2 is an approximation of the iterative scheme of Algorithm 4.1 when the regularization parameter $\gamma$ is large enough. (2) If there is no box constraints, the sequence $\{\hat{y}_b^k\}$ is degenerated to the sequence $\{\hat{z}_b^k\}$. (3) It is easy to see that all the matrices in Algorithm 4.2 could be calculated in advance, which remain invariant during the iterations. Actually, the LU decomposition of the matrices $\kappa(M_1 + M_2) + \rho A^T A$ and $M_1 + \tau M_2 + \rho B^T B$ are also invariant in the ADMM iterations, which can be pre-computed once before the iterative process. (4) The computation efficiency of ADMM relies heavily on the choices of the parameter $\rho$. In this paper, we only choose a fixed constant $\rho = 1$ in the numerical experiment to test the efficiency of our algorithms, please to see [13] for the efficient parameter adaptive selection method.
5 Numerical experiment

In this section, we present three numerical simulations to verify the efficiency of our proposed method. In the light of the convergence analysis in Theorem 4.1, we consider the ADMM iterative error and the edge element discretization error.

In order to obtain an exact optimal solution, we follow the examples in [32] by including a shift control $u_d$ in the objective function (3.1), where $u_d \in H_0(\text{curl})$ is the given data satisfying $\text{div} u_d = 0$. After replacing the term $\frac{\kappa}{2} \| u - u_d \|_{H(\text{curl})}$, we consider the stationary Maxwell’s equation optimal control problem (3.1)-(3.2) defined on $\Omega = [-1,1]^3$. The parameters in the objective function (3.1) are taken as $\tau = 1$ and $\gamma = 10^6$. The exact state variable $y_d$ is given according to equation (3.2) when the control $u_d \in H_0(\text{curl})$ is given, and $y_e$ is determined by $\text{curl} y_d$. It’s not hard to get that the optimal solution of (3.1)-(3.2) for any $\gamma > 0$ is given by $(u_\gamma, y_\gamma) = (u_d, y_d)$ when $y_d \in [y_a, y_b]$.

Example 5.1. The regularization parameter $\kappa$ in the objective function (3.1) is set to 0.1, and the coefficients of the stationary Maxwell’s equation constraint (3.2) are chosen as $\alpha = 1$ and $\beta = 1$. The optimal control variable $u_d$ is given by

$$u_d = \begin{pmatrix}
(x^2_2-x_2)(x^2_3-x_3) \\
(x^2_3-x_3)(x^2_1-x_3) \\
(x^2_1-x_1)(x^2_2-x_2)
\end{pmatrix},$$

which belongs to $H_0(\text{curl})$. Let the box constraints $[y_a, y_b] = [-100, 100]$, which is equivalent to the unconstrained case because that the range of box constraints is large enough.

Fig. 1(a) shows the ADMM errors $\| F(u^k_{\text{ff}, y^k_{\text{ff}}}) - F(u^k_{\text{ff}, \tilde{u}^k_{\text{ff}}}) \| = \| F(y^k_{\text{ff}}, u^k_{\text{ff}}) - F(y^k_{\text{ff}}, \tilde{u}^k_{\text{ff}}) \|$ in Algorithm 4.2 with respect to the $k$-iteration in log-scale when $h = \frac{1}{8}$. From Fig. 1(a), we can find that the ADMM converges much faster than $O(\frac{1}{k})$ as shown in Theorem 4.1, and the error of the ADMM can be close to $10^{-3}$ with 100 iterations. Moreover, Fig. 1(b) shows the ADMM errors in $\| \cdot \|_H$-norm (cf. [37]), which is given by

$$\left\| \frac{\tilde{y}^k_h - y^k_h}{A^k - \bar{A}^k} \right\|_H$$

with $H = \begin{bmatrix}
\rho A^T A & 0 \\
0 & \frac{1}{p} I_N
\end{bmatrix}$. (5.1)

From the Fig. 1(b), we can also find that the numerical convergent order of the ADMM is much faster than $O(\frac{1}{h})$. Fig. 2 gives the images of the three components of numerical solution $\tilde{u}^k_h$ by Algorithm 4.2 with $k = 100$ and $h = \frac{1}{8}$.

In order to verify the convergent order of edge element method, we set $k = \frac{1}{h}$ with $h = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}$ respectively. Fig. 3 shows that the FEM converges of order one.

Example 5.2. All the parameters and functions are chosen the same as in Example 5.1 except for the box constraint. Here, $[y_a, y_b] = [-10^{-2}, 10^{-2}]$. It is easy to verify that the
Figure 1: The ADMM errors by Algorithm 4.2 for Example 5.1: (a) the error $|F(\tilde{u}_{kh}^k, \tilde{y}_{kh}^k) - F(y_{kh}^k, \tilde{u}_{kh}^k)|$; (b) the error in $H$-norm defined by (5.1).

Figure 2: The three components of the numerical solution $\tilde{u}_{kh}^k$ by Algorithm 4.2 with $k = 100$ and $h = \frac{1}{8}$ for Example 5.1: (a) $(\tilde{u}_{kh}^k)_1$ component; (b) $(\tilde{u}_{kh}^k)_2$ component; (c) $(\tilde{u}_{kh}^k)_3$ component.

Figure 3: The FEM errors by Algorithm 4.2 for Example 5.1 with $k = \frac{1}{h}$.
The objective function error of ADMM in log scale

The H−norm error of ADMM in log scale

Figure 4: The ADMM errors by Algorithm 4.2 for Example 5.2: (a) the error $|\mathcal{F}(u^*_h, y^*_h) - \mathcal{F}(\tilde{y}^k_h, \tilde{u}^k_h)|$; (b) the error in $H$-norm defined by (5.1).

Figure 5: The three components of the numerical solution $\tilde{u}^k_h$ by Algorithm 4.2 with $k = 100$ and $h = \frac{1}{8}$ for Example 5.2: (a) $(\tilde{u}^k_h)_1$ component; (b) $(\tilde{u}^k_h)_2$ component; (c) $(\tilde{u}^k_h)_3$ component.

optimal solution is also $(u_d, y_d)$, but we need the projection step to obtain the iteration schemes in Algorithm 4.2.

Fig. 4 shows the ADMM error $|\mathcal{F}(u^*_h, y^*_h) - \mathcal{F}(\tilde{y}^k_h, \tilde{u}^k_h)|$ and the error (5.1) in $H$-norm by Algorithm 4.2 with respect to the $k$-iteration in log-scale when $h = \frac{1}{8}$. From Fig. 4, we can find that the ADMM converges much faster than $O\left(\frac{1}{k}\right)$ as shown in Theorem 4.1. Fig. 4 also confirms that the error curves for the box constraint case are similar to the unconstraint case in Fig. 1, which means that the ADMM algorithm is an efficient method for the constrained optimal control problems. Fig. 5 gives the images of the three components of numerical solution $\tilde{u}^k_h$ by Algorithm 4.2 with $k = 100$ and $h = \frac{1}{8}$.

**Example 5.3.** In this example, the regularization parameter $\kappa$ in the objective function (3.1) is set to $10^{-4}$, and the coefficients in (3.2) are chosen as $\alpha = 100$ and $\beta = 1$ where there
Figure 6: The ADMM errors by Algorithm 4.2 for Example 5.3 with different optimal control variables: (a) case I; (b) case II.

is a jump between two coefficients. We consider the following two different optimal control variables $u_d$:

(I) $u_d = \begin{pmatrix} \sin 2\pi x_2 \sin 2\pi x_3 \\ \sin 2\pi x_1 \sin 2\pi x_3 \\ \sin 2\pi x_1 \sin 2\pi x_2 \end{pmatrix}$,  
(II) $u_d = \begin{pmatrix} \sin 2\pi x_2 \sin 2\pi x_3 \\ \sin 2\pi x_1 \sin 2\pi x_3 \\ (x_1^2-x_1)(x_2^2-x_2) \end{pmatrix}$.

Let box constraints be $[y_a, y_b] = [-10^{-4}I, 10^{-4}I]$.

Fig. 6 shows the ADMM error $|F(u^*_{kh}, y^*_{kh}) - F(\tilde{y}^k_{kh}, \tilde{u}^k_{kh})|$ by Algorithm 4.2 with respect to the $k$-iteration in log-scale. From Fig. 6, we can find that the ADMM converges much faster than $O(1/k)$ in both cases. The error of the ADMM is less than $10^{-3}$ within 100 iterations, where the error of ADMM is comparable to the discretization error of the edge element method. The numerical simulation shows that the ADMM algorithm is efficient even when the coefficients have a large gap.

In order to verify the convergent order of edge element method, we set $k = \frac{1}{h}$ with $h = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32},$ and $\frac{1}{64}$ respectively. Fig. 7 shows that the FEM converges of order one.

6 Conclusions

In this paper, we present an alternating direction method of multipliers to solve the optimal control problem of stationary Maxwell’s equations. The model problem is discretized by the edge element method, and the convergence analysis is established in the form of the objective function error. For the resulting finite dimensional problem, an alternating direction method of multipliers (ADMM) is proposed based on the separable structure. The global convergence analysis is established in the form of the objective function error, which includes the discretization error by the edge element and the iterative error.
by ADMM. Finally, numerical simulations are presented to verify the efficiency of the proposed algorithm. Inspired by [33, 35], we shall combine linearization techniques and ADMM to solve the nonlinear magnetostatic equation optimization problems or the more important nonlinear evolution case in the future.

Acknowledgments

The work of K. Zhang is supported in part by China Natural National Science Foundation (91630201, U1530116, 11471141, 11771179, 11726102), and by the Key Laboratory of Symbolic Computation and Knowledge Engineering of Ministry of Education, Jilin University (93K172018Z01). The work of H. Song is supported in part by China Natural National Science Foundation (11701210), and the education department science and technology project of Jilin Province (JJKH20180113KJ). The authors also wish to thank the High Performance Computing Center of Jilin University and Computing Center of Jilin Province. support.

Appendix: The proof of Corollary 3.1

Proof. Firstly, we prove the estimate (3.10). By the definitions of the operator $S$ and $S_h$, and noting that $V_h \subset H_0(\text{curl})$, we have

$$\|y_\gamma - y_{\gamma h}\|_{H(\text{curl})} = \|Su_\gamma - S_hu_{\gamma h}\|_{H(\text{curl})} \leq \|Su_\gamma - Su_{\gamma h}\|_{H(\text{curl})} + \|Su_{\gamma h} - S_hu_{\gamma h}\|_{H(\text{curl})}$$

$$\leq C\|u_\gamma - u_{\gamma h}\|_{H(\text{curl})} + C_2h^{\frac{1}{2}+\epsilon}\|u_{\gamma h}\|_{H(\text{curl})}$$
The boundedness of the operator $S$, the estimate (3.8), Corollary 4.2 in [32], and the assumption $h < 1$ are used in above proof.

Now, we prove the estimate (3.11). It follows from the definition (3.1) that

$$
\mathcal{J}_1(y_\gamma, u_\gamma) - \mathcal{J}_1(y_{\gamma h}, u_{\gamma h}) = \frac{1}{2}(\|y_\gamma - y_d\|^2 - \|y_{\gamma h} - y_d\|^2) + \frac{\tau}{2}(\|\text{curl} y_\gamma - y_c\|^2 - \|\text{curl} y_{\gamma h} - y_c\|^2)
+ \frac{\kappa}{2}(\|u_h\|^2_{H(\text{curl})} - \|u_{\gamma h}\|^2_{H(\text{curl})})
+ \frac{\tau}{2}(\|\max(y_a - y_{\gamma})\|^2 - \|\max(y_a - y_{\gamma h}, 0)\|^2)
+ \frac{\tau}{2}(\|\max(0, y_\gamma - y_h)\|^2 - \|\max(0, y_{\gamma h} - y_h)\|^2)
:= \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3. \tag{A.2}
$$

By simple calculations, we can derive the following estimates one by one:

$$
\mathcal{J}_1 := \frac{1}{2}(\|y_\gamma - y_d\|^2 - \|y_{\gamma h} - y_d\|^2) + \frac{\tau}{2}(\|\text{curl} y_\gamma - y_c\|^2 - \|\text{curl} y_{\gamma h} - y_c\|^2)
= \frac{1}{2}(y_\gamma + y_{\gamma h} - 2y_d, y_\gamma - y_{\gamma h}) + \frac{\tau}{2}(\text{curl} y_\gamma + \text{curl} y_{\gamma h} - 2y_c, \text{curl} y_\gamma - \text{curl} y_{\gamma h})
\le \frac{1}{2}(\|y_{\gamma h} - y_\gamma\|^2 + 2\|y_\gamma\|^2 + 2\|y_d\|) \|y_\gamma - y_{\gamma h}\|
+ \frac{\tau}{2}(\|\text{curl} (y_{\gamma h} - y_\gamma)\|^2 + 2\|\text{curl} y_\gamma\|^2 + 2\|y_c\|) \|\text{curl} (y_\gamma - y_{\gamma h})\|
\le \frac{1}{2}(1 + \tau)(\|y_{\gamma h} - y_\gamma\|^2_{H(\text{curl})} + 2\|y_\gamma\|^2_{H(\text{curl})} + \|y_d\|^2 + \tau\|y_c\|^2) \|y_\gamma - y_{\gamma h}\|^2_{H(\text{curl})}, \tag{A.3}
$$

$$
\mathcal{J}_2 := \frac{\kappa}{2}(\|u_h\|^2_{H(\text{curl})} - \|u_{\gamma h}\|^2_{H(\text{curl})})
= \frac{\kappa}{2}(u_\gamma + u_{\gamma h}, u_\gamma - u_{\gamma h})_{H(\text{curl})}
\le \frac{\kappa}{2}(\|u_{\gamma h}\|^2_{H(\text{curl})} + 2\|u_h\|^2_{H(\text{curl})}) \|u_\gamma - u_{\gamma h}\|^2_{H(\text{curl})}, \tag{A.4}
$$
\[ J_3 := \frac{\gamma}{2} \left( \| \max(y_d - y_{\gamma}, 0) \|^2 - \| \max(y_a - y_{\gamma h}, 0) \|^2 \right) \]
\[ + \frac{\gamma}{2} \left( \| \max(0, y_{\gamma} - y_b) \|^2 - \| \max(0, y_{\gamma h} - y_b) \|^2 \right) \]
\[ = \frac{\gamma}{2} \left( \max(y_d - y_{\gamma}, 0) + \max(y_a - y_{\gamma h}, 0) \right) \max(y_a - y_{\gamma h}, 0) - \max(y_d - y_{\gamma h}, 0) \]
\[ + \frac{\gamma}{2} \left( \max(0, y_{\gamma} - y_b) + \max(0, y_{\gamma h} - y_b) \right) \max(0, y_{\gamma} - y_b) - \max(0, y_{\gamma h} - y_b) \]
\[ \leq \frac{\gamma}{2} \left( \| y_{\gamma h} - y_{\gamma} \|_y + 2 \| y_{\gamma} \|_y + 2 \| y_b \| \| y_{\gamma} - y_{\gamma h} \| \right) \]
\[ + \frac{\gamma}{2} \left( \| y_{\gamma h} - y_{\gamma} \|_y + 2 \| y_{\gamma} \|_y + 2 \| y_b \| \| y_{\gamma} - y_{\gamma h} \| \right). \]  
\text{(A.5)}

From the assumption that \( h < 1 \), Lemma 3.1, and the estimate (3.10), we obtain
\[ \| u_{\gamma} - u_{\gamma h} \|_{H(\text{curl})} \leq C_1 \| u_{\gamma} \|_{H^{1/2} + (\text{curl})} + \| y_{\gamma} - y_{\gamma h} \|_{H(\text{curl})} \leq C_1 \| u_{\gamma} \|_{H^{1/2} + (\text{curl})}. \]  
\text{(A.6)}

Substituting the estimates (A.3)-(A.6) into (A.2), we obtain
\[ \left| J_\gamma(y_{\gamma}, u_{\gamma}) - J_\gamma(y_{\gamma h}, u_{\gamma h}) \right| \]
\[ \leq \frac{1}{2} (1 + \tau + 2\gamma) (C_1 \| u_{\gamma} \|_{H^{1/2} + (\text{curl})} + 2 \| y_{\gamma} \|_{H(\text{curl})} \| y_{\gamma} - y_{\gamma h} \|_{H(\text{curl})}) \]
\[ + (\| y_d \|_{\gamma} + \| y_c \|_{\gamma} + \gamma \| y_a \|_{\gamma} + \gamma \| y_b \|_{\gamma}) \| y_{\gamma} - y_{\gamma h} \|_{H(\text{curl})} \]
\[ + \frac{\gamma}{2} \left( C_1 \| u_{\gamma} \|_{H^{1/2} + (\text{curl})} + 2 \| u_b \|_{H(\text{curl})} \right) \| u_{\gamma} - u_{\gamma h} \|_{H(\text{curl})}, \]

which together with Lemma 3.1 and estimate (A.1) implies the conclusion.

References


[30] Y. Xu and J. Zou, A convergent adaptive edge element method for an optimal control prob-


