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A Monotone Finite Volume Scheme with Second Order Accuracy for Convection-Diffusion Equations on Deformed Meshes

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Abstract. In this paper, we present a new monotone finite volume scheme for the steady state convection-diffusion equation. The discretization of diffusive flux [33] is utilised and a new corrected upwind scheme with second order accuracy for the discretization of convective flux is proposed based on some available informations of diffusive flux. The scheme is locally conservative and monotone on deformed meshes, and has only cell-centered unknowns. Numerical results are presented to show that the scheme obtains second-order accuracy for the solution and first-order accuracy for the flux.

AMS subject classifications: 35J15, 35J25, 65M08

Key words: Convection-diffusion equation, nonlinear, monotone, deformed meshes.

1 Introduction

It is widely recognized that the discrete maximum principle [11, 30] plays an important role in proving the existence and uniqueness of discrete solution, enforcing numerical stability, and deriving convergence (a priori error estimates) for a sequence of approximate solutions [23]. However, it is not very easy to construct a scheme satisfying the discrete maximum principle on distorted meshes for convection-diffusion equation, especially in the case that the magnitude of convection velocity is much larger than the diffusive coefficient. In this paper, we consider monotonicity (i.e. positivity-preserving) that it can only guarantee nonnegative bound of the numerical solution.

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Many works have been proposed to improve the monotonicity of diffusion scheme, e.g., some restrictive conditions on meshes and diffusive coefficients are given in [12, 21, 24], and some preprocessing or postprocessing methods are proposed in [2, 8, 36]. Recently, some nonlinear schemes without these restrictive conditions have been constructed to guarantee monotonicity in [10,18,19,26,27,29,31–33,38].

Compared with diffusive term, some distinctive features have to be considered for the discretization of convective term such as upwind characteristic. As far as we know, gradient reconstruction [5,6,13,15,20,22,35,37] is one of the most popular method, where the convective flux can be approximated via the upwind approach [3] and controlled by different slope limiting techniques [6, 7, 9, 17]. In [5], a MUSCL-like cell-centered finite volume method is proposed to approximate the solution of steady advection-diffusion equations. The method is based on a least square reconstruction of the vertex values. Moreover, the slope limiter, which is required to prevent the formation and growth of spurious numerical oscillations, is designed to guarantee the existence of the discrete solution of the nonlinear scheme. In [20], a second order accurate monotone finite volume method of the steady-state advection-diffusion equation is presented, which uses a second-order upwind method with a specially designed minimal nonlinear correction for the discretization of advective flux. In [35], a monotone finite volume scheme is also presented, in which the approximation of the advective flux is based on the second-order upwind method with slope limiter. In [40], the integral mean of the cell is used to reconstruct the gradient and the slope technique [6] is used to control the numerical oscillations, however, the accuracy of these methods is lost in some cases of the diffusive coefficient being discontinuous.

In this paper, we focus on the discretization of convective flux. We know that the standard upwind scheme has only first-order accuracy. Hence, in order to improve the accuracy, a corrected method with second order accuracy is proposed. The main feature of the method is that our scheme achieves second order accuracy with a more simpler construction procedure than those in [20, 35, 40]. Moreover, it is efficient for some large deformed meshes, such as Kershaw mesh. It is also efficient for the problem with discontinuous, anisotropic and heterogeneous full tensor coefficients on deformed meshes.

The article is organized as follows. We describe the problem and introduce some notations in Section 2. The discretization of diffusive flux is given in Section 3 and the discretization of convective flux is given in Section 4. In Section 5, we show that our scheme is monotone, and give the detailed algorithm. In Section 6, some numerical results are presented to illustrate the features of our scheme. At last, some conclusions are given in Section 7.

2 The problem

Consider the stationary convection-diffusion problem for unknown function $u=u(x)$: