Numerical Analysis of an Optimal Control Problem Governed by the Stationary Navier-Stokes Equations with Global Velocity-Constrained

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Abstract. A state-constrained optimal control problem governed by the stationary Navier-Stokes equations is studied. Finite element approximation is constructed, the optimal-order a priori $H^1$-norm and $L^2$-norm error estimates are given, for which the optimal state is a nonsingular solution of the Navier-Stokes equations to the optimal control.

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1 Introduction

Fluid flow exists various aspects of human activities, it is an important object of study of mechanics. From 1960s, fluid mechanics began permeating to other disciplines, and they were keeping intersecting with each other, which forms many new cross-disciplinary and interdisciplinary. As the impact of fluid mechanics expanding to many areas, people often encounters such problem in engineering applications: how to control the external conditions of the flow fields such as the temperature, the volume force or the boundary conditions to obtain or get closest to the flow field which has the observed velocity, pressure and boundary. That type of flow control problem becomes a more and more active field, which attracts the interest of researches in the recent years.

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Let us list some theoretical work existing in the literature. Abergel and Temam derive the first optimality conditions and give a gradient algorithm for the control problem of time-dependent Navier-Stokes (N-S) flow in [1]. The state-constrained flow control problems of unsteady N-S flow in the three-dimensional space are studied in [31]. Some theoretical works of the point-wise state-constrained flow control problem are discussed in [11–13], in which the optimality conditions, the regularity of the multiplier and the Lipschitz-stability of disturbance about the constraint set are investigated. In addition, some researchers study the more difficult Cahn-Hilliard-Navier-Stokes system, see [22], etc.

It is obvious that an efficient numerical analysis is quite essential to many applications of flow control. And the finite element method is undoubtedly the most appropriate tool to compute flow control problems. Recently some results of finite element approximation to control problems are developed. Gunzburger, Hou and Svobodny [18–20] study the finite element approximation of stationary N-S flow control problems with some weak control constraints, in which they assume the corresponding linearized system of the first-order optimality conditions defines some isomorphism. Casas and his colleagues discuss the point-wised control constrained stationary N-S flow control problem, and derive the error estimates by using the sufficient optimality conditions in [7]. Chang [8] give the superconvergence of the finite element approximation of the stationary Benard type under the pointwise control constraint. Liu and Yan [24] study the a posteriori error estimates of Stokes flow control problem. On the point of view of numerical algebra, the preconditioner for the N-S flow control model is given in [27]. In addition, a type of topology optimization problems for unsteady N-S flows is studied in [14] and references cited therein. And an approach to shape optimization problem governed by the stationary N-S equations is studied in [4], which is based on transformation to a reference domain with continuous adjoint computations. In addition, Gong and Zhou consider finite element approximations of parabolic control problems and convection-dominated diffusion control problems [16, 32, 33].

Very recently, some people are concerned to study the theory of the state constrained optimal control problems governed by partial differential equations, see, for example, [11, 13], in which they study the state pointwise constrained problem and the mixed control-state constrained one. In fact, one may concentrate on control problems with some weaker constraint conditions on the state in engineering applications. And there have some developments of the numerical analysis for the state integral constrained and the state energy-norm constrained problems, see [6, 25, 26]. But, to the best knowledge of the author, there is no result for the numerical analysis of the state-constrained optimal control problems governed by the stationary N-S equations existing in the literature. This paper is to study the optimal control problem with some global velocity constraints. We pay attention to the numerical analysis of the finite element approximation and concentrate on deriving the optimal-order error estimates of the optimal control and the corresponding states.

The outline of this article is as follows. In Section 2, we state the model problem
and give some general preliminaries of the optimal control problems governed by the stationary N-S equations. In Section 3, we study the first and the second order optimal conditions of the model problem and its finite element approximation. In Section 4, we derive the a priori error estimates for the finite element approximation. Finally, we present some numerical experiments in Section 5 to confirm the theoretical results of the a priori error estimates given in Section 4.

2 Model problem and preliminary results

Let $\Omega$ be a bounded convex subset of $\mathbb{R}^d$, $d = 2, 3$. Throughout this paper we use the standard notations for Sobolev spaces and the corresponding norms, seminorms. For example, $H^m(\Omega)$ denotes the function space $W^{m,2}(\Omega)$ defined on domain $\Omega$ as the usual notation in [2], where $m$ is an integer, $L^2(\Omega) = H^0(\Omega)$ and $H^1_0(\Omega) = \{ v \in H^1(\Omega): v = 0, a.e. on \partial \Omega \}$. Let us denote the $L^2$-inner product by

$$ (v, w) = \int_{\Omega} vw, \quad \forall v, w \in L^2(\Omega), $$

the seminorm and norm in $W^{k,p}(\Omega)$ by $| \cdot |_{k,p,\Omega}$ and $\| \cdot \|_{k,p,\Omega}$, respectively. A boldface letter denotes a vector function or a vector function space, such as $w \in H^1_0(\Omega)$, which means that

$$ w = (w_1, \cdots, w_d)^T, \quad w_i \in H^1_0(\Omega), \quad i = 1, \cdots, d, $$

and the corresponding seminorm and norm in $W^{k,p}(\Omega)$ is denoted by

$$ |w|_{k,p,\Omega} = \left\{ \left( \sum_{i=1}^d |w_i|^p \right)^{\frac{1}{p}} \right\}^\frac{1}{p}, \quad \|w\|_{k,p,\Omega} = \left\{ \sum_{i=1}^d \|w_i\|^p \right\}^{\frac{1}{p}}. $$

The bracket $[ \cdots ]$ contains a group of functions belong to some different function spaces, for example,

$$ [w, q] \in H^1_0(\Omega) \times L^2(\Omega) \leftrightarrow v \in H^1_0(\Omega), \quad q \in L^2(\Omega). $$

We shall use $c, C$ to denote some constant numbers which are independent with the mesh diameter in this paper.

2.1 Model problem

In this paper we investigate the following optimal control problem

$$\begin{align*}
\min_u \quad & J(y, u) = \frac{1}{2} \int_\Omega |y - y_d|^2 + \frac{\alpha}{2} \int_\Omega |u|^2 \\
\text{subject to} \quad & u \in U,
\end{align*}\tag{2.1}$$
subjected to the stationary N-S equations

\[
\begin{align*}
-\nu \Delta y + (y \cdot \nabla)y + \nabla p &= f + u & \text{in } \Omega, \\
\nabla \cdot y &= 0 & \text{in } \Omega, \\
y &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

and

\[
F_i(y) \leq 0, \quad 1 \leq i \leq M,
\]

where \(\Omega\) is a bounded convex polyhedral domain in \(\mathbb{R}^d\) and the constant number \(\alpha > 0\), for the positive integer \(M\) functional \(F_i(\cdot)\) is continuous in \(L^\infty(\Omega)\) for any \(1 \leq i \leq M\). Here \(u \in L^2(\Omega)\) is the control, \(y\) is velocity vector and \(p\) is pressure, the viscosity number \(\nu > 0\).

The constraints on \(y\) are stated in the generalized functionals form, in fact they may be the integral or the \(L^p\)-norm in the real life applications, etc., and we suppose that

\[
\begin{align*}
|F_i(y_1) - F_i(y_2)| + |F_i''(y_1)y_3 - F_i''(y_2)y_3| &\leq C(|y_1|_{0,\infty,\Omega} + |y_2|_{0,\infty,\Omega} + |y_3|_{0,\infty,\Omega}) |y_1 - y_2|_{0,2,\Omega}, \\
1 \leq i \leq M.
\end{align*}
\]

### 2.2 Notations and preliminary results

Let us introduce the following function spaces

\[
Y = H_0^1(\Omega), \quad M = \{L^2(\Omega) / \mathbb{R}\}, \quad U = L^2(\Omega), \quad V_0^2 = \{v \in L^2(\Omega) | \nabla \cdot v = 0\}.
\]

It is clear that \(Y\) is a linear closed subspace of the Hilbert space \(H^1(\Omega)\), and so does the quotient space \(M\) of the Hilbert space \(L^2(\Omega)\), from which we know that \(Y\) and \(M\) are both reflexible. Let us denote by \(H^{-1}(\Omega)\), \(M^*\) the conjugate spaces of \(Y\), \(M\), respectively, and introduce the following operators

\[
\begin{align*}
A: Y &\to H^{-1}(\Omega) \quad \text{s.t.} \quad Aw = -\nu \Delta w, & \forall w \in Y, \\
D: Y &\to H^{-1}(\Omega) \quad \text{s.t.} \quad D(w) = (w \cdot \nabla)w, & \forall w \in Y, \\
B: Y &\to M^* \quad \text{s.t.} \quad Bw = \nabla \cdot w, & \forall w \in Y, \\
B^*: M &\to H^{-1}(\Omega) \quad \text{s.t.} \quad B^*q = \nabla q, & \forall q \in M,
\end{align*}
\]

the bilinear forms and trilinear form

\[
\begin{align*}
(w, z) &= \sum_{i=1}^{d} (w_i, z_i), & \forall w, z \in U, \\
(a(w, z) &= \nu \int_{\Omega} \nabla w : \nabla z = \nu \sum_{i=1}^{d} (\nabla w_i, \nabla z_i), & \forall w, z \in Y, \\
b(w, q) &= -\nabla \cdot (w q), & \forall w \in H^1(\Omega), q \in L^2(\Omega), \\
d(z, w, v) &= ((z \cdot \nabla) w, v), & \forall z, w, v \in H^1(\Omega).
\end{align*}
\]
Moreover, if \( u \) Lemma 2.2. The following properties are well known

\[
\exists \beta_0 > 0: \quad \forall q \in M \quad \exists v \in Y \quad \text{s.t.} \quad \beta_0 \|q\|_{0,2;\Omega} \leq \sup_{v \in Y} \frac{b(w,q)}{\|v\|_{1,2;\Omega}}.
\]  

(2.4)

The following lemma follows from Green’s formula.

**Lemma 2.1.** The operator \( D'(\cdot) \in \mathcal{L}(Y \times Y, H^{-1}(\Omega)) \), \( D'(\cdot)^* \in \mathcal{L}(Y \times Y, H^{-1}(\Omega)) \) satisfies

\[
\langle D'(w)z, v \rangle = d(w, v, z) + d(z, w, v), \quad \langle D'(w)^* z, v \rangle = d(w, v, z) + d(v, w, z),
\]

for all \( v, w, z \in Y \). Especially for all \( w \in V^0_0 \) there holds

\[
\langle D'(w)^* z, v \rangle = ((\nabla w^T, z), v) - d(w, v, z).
\]

Moreover, \( D''(w) \in \mathcal{L}(Y \times Y, H^{-1}(\Omega)) \) for all \( w \in Y \) satisfies

\[
\langle D''(w)(w_1, w_2), z \rangle = d(w_1, w_2, z) + d(w_2, w_1, z) \quad \forall w_1, w_2, z \in Y.
\]

If triplet \([y, p, u]\) satisfies Eq. (2.2), then there is the following well-known regularity result, see, for example, [15, 17, 29] or [7, Theorem 2].

**Lemma 2.2.** If \([y, p, u]\) satisfies (2.6), then there exists a constant \( C > 0 \) such that

\[
\|y\|_{1,2;\Omega} + \|p\|_{0,2;\Omega} \leq C(\|f\|_{-1,\Omega} + \|u\|_{-1,\Omega}).
\]

Moreover, if \( u, f \in U \), then \([y, p] \in H^2(\Omega) \times H^1(\Omega) \) and

\[
\|y\|_{2,2;\Omega} + \|p\|_{1,2;\Omega} \leq C(\|f\|_{0,2;\Omega} + \|u\|_{0,2;\Omega} + \|f\|_{0,2;\Omega}^3 + \|u\|_{0,2;\Omega}^3).
\]

For abbreviation, we introduce operator \( e(\cdot, \cdot) \) from \( Y \times M \) to \( H^{-1}(\Omega) \times M^* \) defined by

\[
e(y, p) = [Ay + D(y) + B^* p - f, By].
\]

Thus, solving N-S equations (2.2) is equivalent to finding \([y, p] \in Y \times M\) satisfy

\[
e(y, p) = [u, 0] \quad \text{in} \quad H^{-1}(\Omega) \times M,
\]

(2.6)

or the variational form

\[
\langle e(y, p), [w, q] \rangle = (u, w) \quad \forall [w, q] \in Y \times M,
\]

(2.7)
where

$$\langle e(y, p), [w, q] \rangle = a(y, w) + d(y, y, w) + b(w, p) - (f, w) - b(y, q).$$

It is obvious that operator $e(y, p)$ is of $C^\infty$ from $Y \times M$ to $H^{-1}(\Omega) \times M^*$, and its Fréchet-derivative $e_{y, p}(y, p) : Y \times M \mapsto H^{-1}(\Omega) \times M^*$ is defined by

$$e_{y, p}(y, p)(z, r) := [-\nu \Delta z + D'(y)z + B^*r, Bz] \quad \forall [z, r] \in Y \times M.$$

It is well known that (2.6) has a unique solution when $\nu$ is large enough with respect to the right-hand side, see, for instance, [29]. That is a strong assumption, although we are interested in the solutions of (2.6) which are locally unique. If $[y, p]$ is a solution of (2.6) to $u$, let us introduce the following equation defining $[z, r]$ to some $g \in H^{-1}(\Omega)$ by

$$\langle e_{y, p}(y, p)(z, r), [w, q] \rangle = (g, w) \quad \forall [w, q] \in Y \times M. \tag{2.8}$$

In this paper we will pay attention to the nonsingular solutions of Eq. (2.6), which is defined as follows.

**Definition 2.1.** A group function $[y, p]$ is called a nonsingular solution of (2.6), if operator $e_{y, p}(y, p)(\cdot, \cdot)$ defines an isomorphism from $Y \times M$ into $H^{-1}(\Omega) \times M^*$.

The following theorem is a straightforward result of the implicit function theorem, see [5, Lemma 1].

**Theorem 2.1.** Let $[y, p]$ be a nonsingular solution of (2.6) to $u$, then we can choose $\delta_0 > 0$ such that an open ball $B([y, p], \delta_0)$ in $Y \times M$, operator $e_{y, p}(\hat{y}, \hat{p})$ defines an isomorphism from $Y \times M$ to $H^{-1}(\Omega) \times M^*$ for all $[\hat{y}, \hat{p}] \in B([y, p], \delta_0)$. Furthermore, note the homogeneity of the second equation in Eqs. (2.2), there exists an operator $\mathcal{G}$ mapping $B(u, \rho_0) \subset H^{-1}(\Omega)$ to $B(y, \delta_0)$ of class $C^\infty$, such that $\hat{y} = \mathcal{G}(\hat{u})$ with a $\hat{p} \in M$ is the unique solution of Eq. (2.6) in $B([y, p], \delta_0)$ to all $\hat{u} \in B(u, \rho_0)$.

**Remark 2.1.** In fact, according to the above result, if operator $e_{y, p}(y, p)$ defines an isomorphism from $Y \times M$ to $H^{-1}(\Omega) \times M^*$, we have $\mathcal{G}'(u) = (I \ 0) \circ e_{y, p}(y, p)^{-1} \circ (I \ 0)^T$ from the homogeneity of the second equation in (2.2), and if $z = G'(u)g$ is in $Y$ for all $g$ in $H^{-1}(\Omega)$, there exists an $r \in M$ such that

$$e_{y, p}(y, p)(z, r) = [g, 0] \quad \text{in } H^{-1}(\Omega) \times M^*. \tag{2.9}$$

At the same time, recalling that spaces $Y$ and $M$ are both reflexible, we know that operator $e_{y, p}(y, p)^*$ also defines an isomorphism from $Y \times M$ to $H^{-1}(\Omega) \times M^*$, then equation

$$e_{y, p}(y, p)^*(\psi, \pi) = [h, 0] \quad \text{in } H^{-1}(\Omega) \times M^*$$

admits a unique solution to all $h \in H^{-1}(\Omega)$.
Further, if $\phi = \mathcal{G}''(u)(\hat{u}_1, \hat{u}_2)$ belongs to $Y$, then there exists a $\varpi \in M$ and the pair $[\phi, \varpi] \in Y \times M$ such that
\begin{equation}
 e_{y,p}(y,p)(\phi, \varpi) = \left[ - \sum_{i+j=3, 1 \leq i, j \leq 2}^{M-1} D'(\mathcal{G}'(u)\hat{u}_i) \mathcal{G}'(u)\hat{u}_j, 0 \right],
\end{equation}
which implies that
\begin{equation}
 \phi = \mathcal{G}'(u) \left( - \sum_{i+j=3, 1 \leq i, j \leq 2}^{M-1} D'(\mathcal{G}'(u)\hat{u}_i) \mathcal{G}'(u)\hat{u}_j \right).
\end{equation}

Remark 2.2. For such variables in $Y \times M$ introduced above, one can derive that
\begin{equation}
 [z, r], [\psi, \pi], [\phi, \varpi] \in H^2(\Omega) \times H^1(\Omega)
\end{equation}
similarly as in the proof of (2.2) provided $y \in H^1(\Omega)$ and $u, \hat{u}_1, \hat{u}_2, f, g, h \in U$.

3 Optimal conditions and finite element approximation

In this section, we give the first order optimal conditions in Lemma 3.1 of problem (2.1) and the second order ones in Lemma 3.3. Furthermore, the corresponding optimal conditions are discussed in finite element spaces, which are stated in Lemma 3.4. Since this type of problems are studied in [6, Casas], [10, Clarke] and references cited therein, some results are given directly without proof here.

Above all, we suppose that the following assumption about the optimal control and state holds in the rest of this paper.

Assumption 1. If $[\bar{u}, \bar{y}]$ is a solution of problem (2.1), let us assume that there is a $\bar{p} \in M$ such that pair $[\bar{y}, \bar{p}]$ is a nonsingular solution of (2.7) to $\bar{u}$.

Then from the above assumption and Theorem 2.1, we know that Problem $(\mathcal{P})$ is equivalent to
\begin{equation}
 \min_{u \in B(\bar{u}, \rho_0)} J(u) = \frac{1}{2} \int_{\Omega} |\mathcal{G}(u) - y_d|^2 + \frac{\alpha}{2} \int_{\Omega} |u|^2, \quad \mathcal{F}_i(u) \leq 0, \quad 1 \leq i \leq M,
\end{equation}
where $\mathcal{F}_i := F_{i} \circ \mathcal{G}$ for any $1 \leq i \leq M$.

3.1 Optimal conditions

To get the first order necessary optimal conditions to problem (2.1), we consider Lagrange function $L : U \times \mathbb{R}^M$ given by
\begin{equation}
 L(u, \lambda_1, \cdots, \lambda_M) = J(u) + \sum_{i=1}^{M} \lambda_i \mathcal{F}_i(u).
\end{equation}

Firstly, we give the first order necessary optimal conditions as follows.
**Lemma 3.1.** If \( \bar{u} \) is a local solution of (2.1), then there exists a real constant vector \( \bar{\lambda} = (\bar{\lambda}_1, \cdots, \bar{\lambda}_M) \) such that

\[
\begin{align*}
(1) & \quad \bar{\lambda}_i \geq 0, \quad \bar{\lambda}_i F'_i(\bar{u}) = 0, \quad 1 \leq i \leq M, \\
(2) & \quad \frac{\partial L}{\partial u}(\bar{u}, \bar{\lambda}) = 0.
\end{align*}
\]

(3.3)

For ease of exposition we introduce an \( S \) given by

\[
S := [u, y, p, \psi, \tau, \lambda] \quad \text{in space} \quad X := U \times Y \times M \times Y \times M \times \mathbb{R}^M,
\]

which is normed by \( ||S||_X = (||u||^2_{0, \Omega} + ||y||^2_{1, 2, \Omega} + ||p||^2_{1, 2, \Omega} + ||\psi||^2_{1, 2, \Omega} + ||\tau||^2_{0, 2, \Omega} + ||\lambda||^2)^{1/2} \). Then optimal conditions (3.3) is equivalent to that there exists an \( \bar{S} \in X \) satisfies equations

\[
\begin{align*}
(1) & \quad (e(\bar{y}, \bar{p}), [w, q]) = (\bar{u}, w) \quad \forall [w, q] \in Y \times M, \\
(2) & \quad (e_{y, p}(\bar{y}, \bar{p})^* (\bar{\psi}, \bar{\tau}), [w, q]) = \left( \bar{y} + \sum_{i=1}^M \bar{\lambda}_i F'_i(\bar{y}) - y_d, w \right) \quad \forall [w, q] \in Y \times M, \\
(3) & \quad \bar{\psi} + a \bar{u} = 0, \\
(4) & \quad \bar{\lambda}_i F_i(\bar{y}) = 0, \quad \bar{\lambda}_i \geq 0, \quad 1 \leq i \leq M.
\end{align*}
\]

Note that \( e_{y, p}(\bar{y}, \bar{p})^* \) is a linear operator, we can also denote that

\[
[\bar{\psi}, \bar{\tau}] = [\bar{\psi}_0, \bar{\tau}_0] + \sum_{j=1}^M \bar{\lambda}_i [\bar{\psi}_j, \bar{\tau}_j],
\]

(3.5)

where the \([\bar{\psi}_j, \bar{\tau}_j] \) satisfy

\[
\begin{align*}
(1) & \quad \langle e_{y, p}(\bar{y}, \bar{p})^* (\bar{\psi}_0, \bar{\tau}_0), [w, q] \rangle = (\bar{y} - y_d, w) \quad \forall [w, q] \in Y \times M, \\
(2) & \quad \langle e_{y, p}(\bar{y}, \bar{p})^* (\bar{\psi}_j, \bar{\tau}_j), [w, q] \rangle = (F'_j(\bar{y}), w) \quad \forall [w, q] \in Y \times M, \quad 1 \leq j \leq M.
\end{align*}
\]

(3.6)

From Assumption 1, we know that operator \( e_{y, p}(\bar{y}, \bar{p}) \) defines an isomorphism from \( Y \times M \) to \( H^{-1}((\Omega) \times M^* \), which means that its adjoint operator \( e_{y, p}(\bar{y}, \bar{p})^* \) defines also an isomorphism from \( Y \times M \) to \( H^{-1}((\Omega) \times M^* \) by Remark 2.1, therefore, the equation

\[
e_{y, p}(\bar{y}, \bar{p})^* (\bar{\psi}, \bar{\tau}) = \left[ \bar{y} + \sum_{i=1}^M \bar{\lambda}_i F'_i(\bar{y}) - y_d, 0 \right]
\]

admits a unique solution \([\bar{\psi}, \bar{\tau}] \) in \( Y \times M \). And it follows from Remark 2.2 that \([\bar{\psi}, \bar{\tau}] \in H^2(\Omega) \times H^2(\Omega) \). At the same time, from the third equation of Eqs. (3.4) we know that the optimal control \( \bar{u} \) belongs to \( H^2(\Omega) \), too. We say that the optimal control \( \bar{u} \) is regular, if there exists a group \( \{ v_j \in H^2(\Omega), \ 1 \leq j \leq M_0 \} \) satisfies

\[
F'_i(\bar{u}) v_j = \delta_{ij} = \begin{cases} 
0, & i \neq j, \\
1, & i = j.
\end{cases}
\]

(3.7)
where the index set $M_0$ is defined as $M_0 = \{ j \mid 1 \leq j \leq M, \ F_j(\bar{u}) = 0 \}$.

Secondly we study the second order optimal conditions of problem (2.1). If $\bar{u}$ is a local solution of (2.1), which implies that $[\bar{u}, \bar{\lambda}]$ is a saddle point of the Lagrange functional defined in (3.2), we have the Taylor expansion of Lagrange functional

$$L(\bar{u} + u, \bar{\lambda}) - L(\bar{u}, \bar{\lambda}) = \left\langle \frac{\partial L}{\partial u}(\bar{u}, \bar{\lambda}), u \right\rangle + \frac{1}{2} \left\langle \frac{\partial^2 L}{\partial u^2}(\bar{u} + tu, \bar{\lambda}), u^2 \right\rangle,$$

for all $\bar{u} + u$ in a neighborhood $O(\bar{u})$. Since operator $G$ is twice continuously Fréchet-differentiable in $O(\bar{u})$, and using the first order optimal condition (3.3)-(1) we have the following necessary second order optimal conditions.

**Lemma 3.2.** If $\bar{u}$ is a local solution of (2.1), then there holds

$$\left\langle \frac{\partial^2 L}{\partial u^2}(\bar{u}, \bar{\lambda}), u^2 \right\rangle \geq 0 \ \forall u \in U.$$

Finally, the gap between the necessary second optimal conditions and the sufficient ones is that whether $\bar{u}$ is a strict local minimal point of problem (2.1), and the second order sufficient optimal conditions are stated as follows.

**Lemma 3.3.** If $\bar{u}$ is a local solution of (2.1), let us assume

$$\left\langle \frac{\partial^2 L}{\partial u^2}(\bar{u}, \bar{\lambda}), u^2 \right\rangle > 0 \ \forall u \in U \setminus \{0\}$$

(3.8) holds. Then there exists a constant number $c_0 > 0$ such that

$$c_0 \| u \|^2_{\Omega} \leq \left\langle \frac{\partial^2 L}{\partial u^2}(\bar{u}, \bar{\lambda}), u^2 \right\rangle$$

(3.9)

holds for all $u$ in $U$.

For more details of the discussion of the optimal conditions the reader may refer to Casas [6] and its references.

### 3.2 Finite element approximation

We only consider $n$-simply element, which is widely used in the finite element approximation. Let $T_h = \bigcup \tau$ be a quasi-regular triangulation of $\Omega$ with maximum mesh size $h := \max_{\tau \in T_h} \{ \text{diam}(\tau) \}$ and $T^U_h = \bigcup \tau_U$ be another partitioning of $\Omega$ with maximum mesh size $h_U := \max_{\tau_U \in T^U_h} \{ \text{diam}(\tau_U) \}$, in which each element has at most one face on $\partial \Omega$, and $\tau$ and $\tau'$ (or $\tau_U$ and $\tau'_U$) have either only one common vertex or a whole edge in 2-d case if $\tau$ and $\tau' \in T_h$ (or $\tau_U$ and $\tau'_U \in T^U_h$).
Associated with $T_h$ are two finite element spaces $Y_h \subset Y$ and $M_h \subset M$ such that there exist two operators $T_h^Y \in \mathcal{L}(Y, Y_h)$, $T_h^M \in \mathcal{L}(M, M_h)$ satisfy

1. $\| w - T_h^Y w \|_{0,2, \Omega} + h \| w - T_h^Y w \|_{1,2, \Omega} \leq C h^{m+1} \| w \|_{m+1,2, \Omega}$ \quad $\forall v \in Y \cap H^{m+1}(\Omega)$,
2. $\| q - T_h^M q \|_{0,2, \Omega} \leq C h^{l+1} \| q \|_{l+1,2, \Omega}$ \quad $\forall q \in M \cap H^{l+1}(\Omega)$, (3.10)
3. $\exists w_h \in Y_h$, $\beta_h^0 > 0$ \quad s.t. $\beta_h^0 \| w_h \|_{1,2, \Omega} \| q_h \|_{0,2, \Omega} \leq b(w_h, q_h)$ \quad $\forall q_h \in M_h$.

The above assumptions are satisfied for the standard Hood-Taylor (P2-P1 that $m=l+1=2$) finite element method and Mini (P1-Bubble,P1 that $m=l=1$) finite element method, see [3, 15, 28], etc.

Associated with $T_h^U$ is another finite element space

$$U_h := \{ u_h \in U, u_h|_{\tau_j} \in P_{m_\Omega}(\tau_j) \quad \forall \tau_j \in T_h^U \} \subset U$$

such that there exists an operator $T_h^U \in \mathcal{L}(U, U_h)$ satisfying (see [9])

$$\| u - T_h^U u \|_{0,2, \Omega} \leq C h^{m_{U1}+1} \| u \|_{m_{U}, 2, \Omega} \quad \forall u \in U \cap H^{m_{U}+1}(\Omega), \quad m_{U} = 0, \ldots, m.$$ (3.11)

The finite element approximation of the optimal control problem $(\mathcal{P}_h)$ can be read as:

$$\begin{align*}
(\mathcal{P}_h) \quad \min_{u \in U_h} & \quad J_h(u) = \frac{1}{2} \int_{\Omega} |G_h(u) - y_d|^2 + \frac{\alpha}{2} \int_{\Omega} |u|^2 \\
\text{s.t.} & \quad F_h(u) = 0, \quad 1 \leq i \leq M,
\end{align*}$$

where $F_h(u) = F_i \circ G_h(u)$ and $y_h = G_h(u)$ such that there exists a $p_h \in M_h$ satisfy

$$\langle \epsilon^h(y_h, p_h), [w_h, q_h] \rangle = (u, w_h) \quad \forall [w_h, q_h] \in Y_h \times M_h.$$ (3.12)

Here

$$\langle \epsilon^h(y_h, p_h), [w_h, q_h] \rangle = a(y_h, w_h) + d(y_h, y_h, w_h) + b(w_h, p_h) - (f, w_h) + b(y_h, q_h).$$

Associated with the above problem we consider the corresponding Lagrange functional $L_h: U_h \times \mathbb{R}^M$ given by

$$L_h(u, \lambda) = J_h(u) + \sum_{i=1}^{M} \lambda_i F_h(u).$$ (3.13)

For simplicity, we denote $S_h = (u_h, y_h, p_h, \psi_h, \pi_h, \lambda_h)$ and $X_h = U_h \times Y_h \times M_h \times Y_h \times M_h \times \mathbb{R}^M \subset X$ in what follows. Similarly we get the first order optimal conditions of problem (3.12).

**Lemma 3.4.** If $\bar{u}_h$ is a local solution of (3.12), then there exists a real constant vector $\bar{\lambda}_h := (\bar{\lambda}_{h1}, \ldots, \bar{\lambda}_{hM})$ such that

1. $\bar{\lambda}_{hi} \geq 0$, $\bar{\lambda}_{hi} F_h(\bar{u}_h) = 0$,
2. $\frac{\partial L_h}{\partial u_h}(\bar{u}_h, \bar{\lambda}_h) = 0$, (3.14)
which is equivalent to that there exists an $\bar{\mathbf{S}}_h \in \mathbf{X}_h$ satisfying equations

\begin{align}
(1) \quad & \langle e_h^I(y_h, \bar{p}_h), [w_h, q_h] \rangle = (\bar{u}_h, w_h) \quad \forall [w_h, q_h] \in \mathbf{Y}_h \times \mathbf{M}_h, \\
(2) \quad & \langle e_{y,p}^I(y_h, \bar{p}_h)^* (\bar{\mathbf{u}}_h, \bar{\mathbf{v}}_h), [w_h, q_h] \rangle = (\bar{y}_h + \sum_{i=1}^{M} \bar{\lambda}_h_i F_{hi}^I(y_h) - y_d, w_h) \quad \forall [w_h, q_h] \in \mathbf{Y}_h \times \mathbf{M}_h, \\
(3) \quad & \mathcal{P}_h \bar{\mathbf{u}}_h + \alpha \bar{u}_h = 0, \\
(4) \quad & \bar{\lambda}_h_i F_{hi}(y_h) = 0, \quad \bar{\lambda}_h_i \geq 0,
\end{align}

(3.15)

where

\begin{align}
\langle e_{y,p}^I(y_h, \bar{p}_h)^* (\bar{\mathbf{u}}_h, \bar{\mathbf{v}}_h), [w_h, q_h] \rangle = & a(\bar{\mathbf{u}}_h, w_h) + d(\mathbf{w}_h, \mathbf{y}_h, \mathbf{q}_h) + d(\mathbf{y}_h, \mathbf{w}_h, \mathbf{q}_h) + b(\mathbf{w}_h, \mathbf{q}_h) - b(\mathbf{q}_h, \mathbf{q}_h),
\end{align}

and $\mathcal{P}_h$ is a $L^2$-projection operator from $\mathbf{U}$ to $\mathbf{U}_h$.

Here, we also introduce the following notations

\[ [\bar{\mathbf{u}}_h, \bar{\mathbf{v}}_h] = [\bar{\mathbf{u}}_h, \bar{\mathbf{v}}_h] + \sum_{j=1}^{M} \bar{\lambda}_h_j [\bar{\mathbf{u}}_h, \bar{\mathbf{v}}_h_j, \bar{\mathbf{w}}_h], \]

(3.16)

where the $[\bar{\mathbf{u}}_h, \bar{\mathbf{v}}_h]$ satisfy

\begin{align}
\langle e_{y,p}^I(y_h, \bar{p}_h)^* (\bar{\mathbf{u}}_h, \bar{\mathbf{v}}_h), [w_h, q_h] \rangle = & (\bar{y}_h - y_d, w_h) \quad \forall [w_h, q_h] \in \mathbf{Y}_h \times \mathbf{M}_h, \\
\langle e_{y,p}^I(y_h, \bar{p}_h)^* (\bar{\mathbf{u}}_h, \bar{\mathbf{v}}_h), [w_h, q_h] \rangle = & (F_{\bar{y}_h}(\bar{y}_h), w_h) \quad \forall [w_h, q_h] \in \mathbf{Y}_h \times \mathbf{M}_h, \quad 1 \leq j \leq M.
\end{align}

(3.17)

Now, let us state the second order necessary optimal conditions.

**Lemma 3.5.** If $\bar{\mathbf{u}}_h$ is a local solution of problem (3.12), and $\bar{\lambda}_h$ is the Lagrange multiplier defined in Lemma 3.4, then they satisfy

\[ \langle \frac{\partial^2 L_h}{\partial \bar{u}_h^2} (\bar{\mathbf{u}}_h, \bar{\lambda}_h) \rangle \geq 0 \quad \forall \bar{\mathbf{u}}_h \in \mathbf{U}_h. \]

4 A priori estimate

To deal with the numerical analysis and get the a priori estimates, we shall study the convergence of the approximation to a nonsingular solution of the nonlinear problem, more precisely speaking, we are concerned to approximate to a local unique solution $\bar{\mathbf{u}}$ of problem (2.1) in $B(\bar{\mathbf{u}}, \rho_0)$.
4.1 Local convergence

First, we give the following convergence of the discrete optimal control.

**Theorem 4.1.** If Assumption 1 holds and \( \{ \tilde{u}_h \} \) is a sequence solution of problem (3.12) corresponding to \( h \), then there exists a subsequence which is still denoted by \( \{ \tilde{u}_h \} \), its weak-limit \( \tilde{u} \) is a solution of problem (2.1) and satisfies that

\[
\lim_{h \to 0} \| \tilde{u} - \tilde{u}_h \|_{0, 2; \Omega} = 0.
\]  

(4.1)

Further, if \( \tilde{u} \) is a regular solution of problem (2.1) and \( \tilde{S}, \tilde{S}_h \) are the corresponding solutions of Eqs. (3.4), (3.15), respectively, then

\[
\lim_{h \to 0} \| \tilde{S} - \tilde{S}_h \|_\chi = 0
\]

(4.2)

holds true.

**Proof.** It is obvious that \( \{ \tilde{u}_h \} \) is uniformly bounded in \( U \), then there exists a subsequence (still denoted by \( \{ \tilde{u}_h \} \)) converges weakly in \( U \), we denote the limit by \( \tilde{u} \). So, if there exists a \( \tilde{v} \) minimize \( \tilde{J}(\cdot) \), there must hold

\[
\tilde{J}(\tilde{u}) = \lim_{h \to 0} \tilde{J}_h(\tilde{u}_h) \leq \lim_{h \to 0} \tilde{J}_h(P_h \tilde{v}) = \lim_{h \to 0} \tilde{J}(P_h \tilde{v}) = \inf_{v \in B(\tilde{u}, 0)} \tilde{J}(v) \leq \tilde{J}(\tilde{u}).
\]

In the meantime, we know \( \tilde{J}_i(\tilde{u}) = \lim_{h \to 0} \tilde{J}_h(\tilde{u}_h) \leq 0 \) holds for any \( 1 \leq i \leq M \), then the weak-limit \( \tilde{u} \) is a solution of problem (2.1).

Observing the Taylor expansion of \( \tilde{J}(\cdot) \) nearby \( \tilde{u} \) in \( B(\tilde{u}, \rho_0) \), we know that there exists \( 0 < s < 1 \) such that

\[
\tilde{J}(\tilde{u}_h) = \tilde{J}(\tilde{u}) + \langle \tilde{J}'(\tilde{u}), \tilde{u}_h - \tilde{u} \rangle + \frac{1}{2} \langle \tilde{J}''(s \tilde{u}_h + (1 - s) \tilde{u}_h), (\tilde{u}_h - \tilde{u})^2 \rangle
\]

\[
= \tilde{J}(\tilde{u}) + \langle \tilde{y} - y_d, G'(\tilde{u})(\tilde{u}_h - \tilde{u}) \rangle + \alpha(\tilde{u}, \tilde{u}_h - \tilde{u}) + \alpha \| \tilde{u}_h - \tilde{u} \|^2_{0, 2; \Omega}
\]

\[
+ \langle \tilde{y} - y_d, G''(s \tilde{u}_h + (1 - s) \tilde{u}_h)(\tilde{u}_h - \tilde{u}) \rangle + \| G'(s \tilde{u}_h + (1 - s) \tilde{u}_h)(\tilde{u}_h - \tilde{u}) \|^2_{0, 2; \Omega},
\]

which implies that

\[
\alpha \| \tilde{u}_h - \tilde{u} \|^2_{0, 2; \Omega} \leq \left\{ \tilde{J}(\tilde{u}_h) - \tilde{J}_h(\tilde{u}_h) + \tilde{J}_h(\tilde{u}_h) - \tilde{J}(\tilde{u}) - \langle \tilde{y} - y_d, G'(\tilde{u})(\tilde{u}_h - \tilde{u}) \rangle
\]

\[
- \alpha(\tilde{u}, \tilde{u}_h - \tilde{u}) - \langle \tilde{y} - y_d, G''(s \tilde{u}_h + (1 - s) \tilde{u}_h)(\tilde{u}_h - \tilde{u})^2 \rangle \right\} \frac{h \to 0}{h \to 0} \to 0,
\]

which is exactly (4.1).
Furthermore, if \( \bar{u} \) is a regular optimal control of problem (2.1), using (2.3), (3.7) and optimality conditions (3.3), (3.14), for \( 1 \leq i \leq M_0 \) we have

\[
\lambda_i - \lambda_{hi} = (\lambda_i - \lambda_{hi}) F_i' (\bar{u}) v_i \\
= \lambda_i F_i' (\bar{u}) v_i - \lambda_{hi} F_i' (\bar{u}) v_i + \lambda_{hi} (F_i' (\bar{u}) v_i - F_i' (\bar{u}) v_i) \\
= (\bar{y} - y_d, G' (\bar{u}) v_i) - (\bar{y}_h - y_d, G_h' (\bar{u}_h) v_i) + (\bar{u} - u_h, v_i) + \lambda_{hi} (F_i' (\bar{u}) v_i - F_i' (\bar{u}) v_i) \\
= (G (\bar{u}) - y_d, G' (\bar{u}) v_i) - (G_h (\bar{u}_h) - y_d, G_h' (\bar{u}_h) v_i) + (\bar{u} - u_h, v_i) \\
+ \lambda_{hi} (F_i' (\bar{u}_h) v_i - F_i' (\bar{u}) v_i) \\
\leq C (\| G (\bar{u}) - G_h (\bar{u}_h) \|_{0,2, \Omega} + \| G' (\bar{u}) v_i - G_h' (\bar{u}_h) v_i \|_{0,2, \Omega} + \| \bar{u} - u_h \|_{0,2, \Omega}) \xrightarrow{h \to 0} 0, 
\]

(4.3)

where we use the standard finite element error estimates for the N-S equations and the linearized equations provided Assumption 1, more details of this type of error estimates of the finite element approximation will be studied in the next subsection, see Lemma 4.5, etc. Similarly, using \( \| \bar{u}_h - \bar{u} \|_{0,2, \Omega} + | \lambda - \lambda_h | \xrightarrow{h \to 0} 0 \) we can easily get inequality (4.2). \( \square \)

And then, using the above theorem, we can derive the following lemma, which is necessary for deriving the optimal-order error estimates.

**Lemma 4.1.** Assume that \( \tilde{S} \) (or \( \tilde{S}_h \)) is a solution of Eqs. (3.4) (or Eqs. (3.15)) and the conditions in Lemmas 3.3, 3.5 and Theorem 4.1 hold, then there exists a positive constant number \( h_0 > 0 \) such that

\[
\langle \frac{\partial^2 L_h}{\partial u^2} (\bar{u}_h, \bar{\lambda}_h, \bar{u}^2) \rangle \geq \frac{C_0}{2} \| u \|_{0,2, \Omega}^2 \quad \forall u \in U_h
\]

holds for \( h, h_U \leq h_0 \).

**Proof.** From (2.10) and Eqs. (3.4), (3.15), for all \( u \in U_h \), we have

\[
\langle \frac{\partial^2 L_h}{\partial u^2} (\bar{u}_h, \bar{\lambda}_h, u^2) \rangle = \langle \frac{\partial^2 L_h}{\partial u^2} (\bar{u}_h, \bar{\lambda}_h, u^2) \rangle + \| G' (\bar{u}) u \|_{0,2, \Omega} - \| G_h' (\bar{u}_h) u \|_{0,2, \Omega} \\
+ (\bar{y} - y_d, G'' (\bar{u}) u^2) - (\bar{y}_h - y_d, G_h'' (\bar{u}_h) u^2) \\
+ \sum_{i=1}^{M} \left\{ \lambda_i \left( F_i'' (\bar{y}) \circ (G' (\bar{u}) u) ^2 + F_i'' (\bar{y}) \circ G'' (\bar{u}) u^2 \right) - \lambda_{hi} F_i'' (\bar{y}_h) \circ (G_h' (\bar{u}_h) u) ^2 + F_i'' (\bar{y}_h) \circ G_h'' (\bar{u}_h) u^2 \right\} \\
\leq \langle \frac{\partial^2 L_h}{\partial u^2} (\bar{u}_h, \bar{\lambda}_h, u^2) \rangle + C \left( \| u \|_{0,2, \Omega} \left( | \bar{\lambda} - \bar{\lambda}_h | \right) \\
+ \| G (\bar{u}) - G_h (\bar{u}_h) \|_{0,2, \Omega} + \| G_h (\bar{u}) - G_h (\bar{u}_h) \|_{0,2, \Omega} \right) \\
+ \| u \|_{0,2, \Omega} \left( \| G' (\bar{u}) u - G_h' (\bar{u}_h) u \|_{0,2, \Omega} + \| G_h' (\bar{u}) u - G_h' (\bar{u}_h) u \|_{0,2, \Omega} \right)
\]

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[Lemma 4.2] Let $\bar{S}(or \bar{S}_h)$ be the solution of Eqs. (3.4) (or Eqs. (3.15)), assume $\bar{u}$ is regular and Assumption 1, conditions in Lemmas 3.3, 3.5 hold, then there exists $\tilde{h} > 0$ such that

$$
\| \bar{y} - \bar{y}_h \|_{1,2,\Omega} + \| \bar{p} - \bar{p}_h \|_{0,2,\Omega} + \| \bar{\phi} - \bar{\phi}_h \|_{1,2,\Omega} + \| \bar{\pi} - \bar{\pi}_h \|_{0,2,\Omega} \leq C(h^m + h^{m+2}) \tag{4.4}
$$

and

$$
\| \bar{u} - \bar{u}_h \|_{0,2,\Omega} \leq C(h^{m+1} + h^{m+1}) \tag{4.5}
$$

hold for $h, h_U < \tilde{h}$ and $u_h \in B(u, \rho_0)$, provided $[\bar{y}, \bar{p}], [\bar{\phi}, \bar{\pi}]$ (for $0 \leq j \leq M$) belong to $H^{m+1}(\Omega) \cap Y$. Furthermore, the $L^2$-estimate for the state, co-state and Lagrange multiplier

$$
\| \bar{y} - \bar{y}_h \|_{0,2,\Omega} + \| \bar{\phi} - \bar{\phi}_h \|_{0,2,\Omega} + \| \phi_h - \bar{u}_h \|_{0,2,\Omega} + \| \bar{\lambda} - \bar{\lambda}_h \| \leq C(h^{m+1} + h^{m+2}) \tag{4.6}
$$

holds true under the same conditions.

Secondly, before giving the proof of the a priori error estimates, we consider the estimates of some general linearized equations, whose results can be used to yield the estimates of finite element approximation of the state and co-state equations, in the proof of Lemmas 4.3-4.4.

[Lemma 4.2] If $[z, r]$ is small in $Y \times M$ and satisfies equation

$$
\langle e_{\gamma, p}^h(\bar{y}, \bar{p}) (z, r), \{w_h, q_h\} \rangle = (v, w_h) \quad \forall [w_h, q_h] \in Y_h \times M_h, \tag{4.7}
$$
then there exists \( h_1 > 0 \) such that

\[
\| z \|_{1,2,\Omega} + \| r \|_{0,2,\Omega} \leq C (\| v \|_{\mathbf{H}^{-1}(\Omega)} + \| z - I_h^Y z \|_{1,2,\Omega} + \| r - I_h^M r \|_{0,2,\Omega})
\]

\[
\| z \|_{0,2,\Omega} \leq C (\| v \|_{\mathbf{H}^{-1}(\Omega)} + h \| z - I_h^Y z \|_{1,2,\Omega} + h \| r - I_h^M r \|_{0,2,\Omega})
\]

(4.8)

hold for \( h < h_1 \). Analogously, if the pair \([\psi, \pi]\) small in \( Y \times M \) is a solution of equation

\[
\langle e_{\gamma,p} (\bar{y}, \bar{\rho})^\star (\psi, \pi), [w_h, q_h] \rangle = (u, w_h) \quad \forall [w_h, q_h] \in Y_h \times M_h,
\]

(4.9)

then there exists \( h_2 > 0 \), the following estimates

\[
\| \psi \|_{1,2,\Omega} + \| \pi \|_{0,2,\Omega} \leq C (\| u \|_{\mathbf{H}^{-1}(\Omega)} + \| \psi - I_h^Y \psi \|_{1,2,\Omega} + \| \pi - I_h^M \pi \|_{0,2,\Omega}),
\]

\[
\| \psi \|_{0,2,\Omega} \leq C (\| u \|_{\mathbf{H}^{-1}(\Omega)} + h \| \psi - I_h^Y \psi \|_{1,2,\Omega} + h \| \pi - I_h^M \pi \|_{0,2,\Omega})
\]

(4.10)

hold for \( h < h_2 \).

**Proof.** In fact, we only need to prove the estimates (4.8), the proof of (4.10) can be derived similarly.

Taking \([w_h, q_h] = [I_h^Y z, I_h^M r]\) in Eq. (4.7), it is easy to get

\[
a(z, I_h^Y z) = (v, I_h^Y z) - d(\bar{y}, z, I_h^Y z) - d(z, \bar{y}, I_h^Y z) - b(I_h^Y z - z, r) + b(z, r - I_h^M r),
\]

accordingly,

\[
v \| z \|_{1,2,\Omega}^2 = a(z, z - I_h^Y z) + a(z, I_h^Y z)
\]

\[
\leq C \| \bar{y} \|_{2,2,\Omega} (\| z \|_{1,2,\Omega} + \| r \|_{0,2,\Omega})
\]

\[
\times (\| v \|_{\mathbf{H}^{-1}(\Omega)} + \| z - I_h^Y z \|_{1,2,\Omega} + \| r - I_h^M r \|_{0,2,\Omega} + \| z \|_{0,2,\Omega}).
\]

(4.11)

On the other hand, by the LBB-condition (3.10)-(3), we know that there exists a \( \hat{w}_h \in Y_h \) such that

\[
\beta_h^0 \| \hat{w}_h \|_{1,2,\Omega} \| I_h^M r \|_{0,2,\Omega} = b(\hat{w}_h, I_h^M r),
\]

then we obtain

\[
\| r \|_{0,2,\Omega} \leq \| r - I_h^M r \|_{0,2,\Omega} + \| I_h^M r \|_{0,2,\Omega} = \| r - I_h^M r \|_{0,2,\Omega} + \frac{1}{\beta_h^0 \| \hat{w}_h \|_{1,2,\Omega}} b(\hat{w}_h, I_h^M r)
\]

\[
= \| r - I_h^M r \|_{0,2,\Omega} + \frac{1}{\beta_h^0 \| \hat{w}_h \|_{1,2,\Omega}} \left\{ (v, \hat{w}_h) - \langle e_{\gamma,p} (\bar{y}, \bar{\rho}) (z, r - I_h^M r), [\hat{w}_h, 0] \rangle \right\}
\]

\[
\leq C (\| \bar{y} \|_{2,2,\Omega} (\| v \|_{\mathbf{H}^{-1}(\Omega)} + \| r - I_h^M r \|_{0,2,\Omega} + \| z \|_{1,2,\Omega}),
\]

which substituted into inequality (4.11) yields

\[
\| z \|_{1,2,\Omega} \leq C (\| v \|_{\mathbf{H}^{-1}(\Omega)} + \| z - I_h^Y z \|_{1,2,\Omega} + \| r - I_h^M r \|_{0,2,\Omega} + \| z \|_{0,2,\Omega}).
\]

(4.12)
and there holds
\[ \|z\|_{1,2,\Omega} + \|r\|_{0,2,\Omega} \leq C (\|v\|_{H^{-1}(\Omega)} + \|z - I_h^y z\|_{1,2,\Omega} + \|r - I_h^M r\|_{0,2,\Omega} + \|z\|_{0,2,\Omega}). \] (4.13)

In the sequel, we need to estimate \( \|z\|_{0,2,\Omega} \). Note that Assumption 1 and Remark 2.1, since the operator \( e_{y,p}(\bar{y}, \bar{p})^* \) defines an isomorphism from \( Y \times M \) to \( H^{-1}(\Omega) \times M^* \), we know that equation
\[ e_{y,p}(\bar{y}, \bar{p})^* (\phi, \omega) = [h, 0] \] (4.14)

admits a unique solution \([\phi, \omega] \in Y \times M\), and from Lemma 2.2 and Remark 2.2, we have
\[ \|\phi\|_{2,\Omega} + \|\omega\|_{1,2,\Omega} \leq C \|h\|_{0,2,\Omega} \quad \forall \|h\|_{0,2,\Omega} \leq 1. \] (4.15)

Thus, taking \( h = z \) and using (4.15), we can deduce that
\[ \|z\|_{0,2,\Omega}^2 = \langle e_{y,p}(\bar{y}, \bar{p})^* (\phi, \omega), (z, r) \rangle \]
\[ = \langle e_{y,p}(\bar{y}, \bar{p}) (z, r), [\phi - I_h^y [\phi, \omega - I_h^M \omega]] - (v, I_h^y \phi) \rangle \]
\[ \leq C \|z\|_{1,2,\Omega} \|r\|_{0,2,\Omega} \quad \forall \|h\|_{0,2,\Omega} \leq 1. \] (4.16)

Therefore, combining inequalities (4.16) with (4.13), we deduce that there exists \( h_1 > 0 \) such that
\[ \|z\|_{1,2,\Omega} + \|r\|_{0,2,\Omega} \leq C (\|z - I_h^y z\|_{1,2,\Omega} + \|r - I_h^M r\|_{0,2,\Omega} + \|v\|_{H^{-1}(\Omega)}) \]
\[ \|z\|_{0,2,\Omega} \leq C (h_1 \|z - I_h^y z\|_{1,2,\Omega} + k \|r - I_h^M r\|_{0,2,\Omega} + \|v\|_{H^{-1}(\Omega)}) \]

holds true for \( h < h_1 \), which are exactly inequalities (4.8). The proof is finished. \[ \square \]

Next, with the above lemma, we can generate the following lemmas about the error analysis of the states and co-states equations and their corresponding finite element approximation.

**Lemma 4.3.** Let \([\bar{y}_h, \bar{p}_h]\) be the solution of Eq. (3.15) and \([\bar{y}, \bar{p}]\) be the solution of Eq. (3.4), then there exists \( h_3 \), for \( h < h_3 \) we have
\[ \|\bar{y} - \bar{y}_h\|_{1,2,\Omega} + \|\bar{p} - \bar{p}_h\|_{0,2,\Omega} \leq C (h^m + h_{\alpha}^{m+2} + \|P_h \tilde{u} - \tilde{u}_h\|_{0,2,\Omega}), \] (4.17)

and
\[ \|\bar{y} - \bar{y}_h\|_{0,2,\Omega} \leq C (h^{m+1} + h_{\alpha}^{m+2} + \|P_h \tilde{u} - \tilde{u}_h\|_{0,2,\Omega}). \] (4.18)

**Proof.** Combining Eqs. (3.4) and (3.15), we get that for all \([w_h, q_h]\) \( \in Y_h \times M_h \) there is
\[ \langle e_{y,p}^h (\bar{y}, \bar{p}) (\bar{y} - \bar{y}_h, \bar{p} - \bar{p}_h), [w_h, q_h] \rangle = \langle \tilde{u} - \tilde{u}_h, w_h \rangle + d(\bar{y} - \bar{y}_h, \bar{y} - \bar{y}_h, w_h). \] (4.19)
Noticing that \( \bar{y}_h = \bar{y}_h, I^M_h \bar{p}_h = \bar{p}_h \), applying the results (4.8) of Lemma 4.2, we then obtain
\[
\| \bar{y}_h - \bar{y}_h \|_{1,2,\Omega} + \| \bar{p} - \bar{p}_h \|_{0,2,\Omega} \\
\leq C \left\{ \| \bar{u} - \mathcal{P}_h \bar{u} \| + \mathcal{P}_h \bar{u} - \bar{u}_h + \| (\bar{y}_h - \bar{y}_h) \cdot \nabla (\bar{y}_h - \bar{y}_h) \|_{H^{-1}(\Omega)} + \| \bar{y}_h - I^M_h \bar{p} \|_{1,2,\Omega} + \| \bar{p} - I^M_h \bar{p} \|_{0,2,\Omega} \right\}
\leq C \left( h^{m+2}_U \| \bar{y} \|_{1,2,\Omega} + \| \bar{p} - \bar{p}_h \|_{0,2,\Omega} \right),
\]
or equivalently
\[
\| \bar{y}_h - \bar{y}_h \|_{1,2,\Omega} + \| \bar{p} - \bar{p}_h \|_{0,2,\Omega} \\
\leq C \left( h^{m+2}_U + \| \mathcal{P}_h \bar{u} - \bar{u}_h \|_{0,2,\Omega} \right),
\]
since the inequality
\[
\frac{1}{2} \left( \| \bar{y}_h - \bar{y}_h \|_{1,2,\Omega} + \| \bar{p} - \bar{p}_h \|_{0,2,\Omega} \right) \leq (1 - C \| \bar{y}_h \|_{1,2,\Omega}) \| \bar{y}_h \|_{1,2,\Omega} + \| \bar{p} - \bar{p}_h \|_{0,2,\Omega}
\]
holds for \( 0 < h < h_3 \) by Theorem 4.1. Analogously, we also have
\[
\| \bar{y}_h - \bar{y}_h \|_{0,2,\Omega} \leq C \left( h^{m+2}_U + 1 + \| \mathcal{P}_h \bar{u} - \bar{u}_h \|_{0,2,\Omega} \right).
\]
The proof is thus finished. \( \square \)

**Lemma 4.4.** For \( 0 \leq j \leq M \) that \( \bar{y}_{hj}, \bar{p}_{hj} \) and \( \bar{y}_j, \bar{p}_j \) defined in Eqs. (3.17) and (3.6), then there exists \( h_4 \) such that
\[
\| \bar{y}_j - \bar{y}_{hj} \|_{1,2,\Omega} + \| \bar{p}_j - \bar{p}_{hj} \|_{0,2,\Omega} \leq C \left( h^m + h^{m+2}_U \right), \]
\[
\| \bar{y}_j - \bar{y}_{hj} \|_{0,2,\Omega} \leq C \left( h^{m+1} + h^{m+2}_U \right),
\]
hold true for \( 0 < h < h_4 \).

**Proof.** From Eqs. (3.6) and (3.17), it is easy to get that
\[
\langle e^{h}_j, \bar{y}_h \rangle = \langle e^{h}_j, \bar{y}_h \rangle - \langle e^{h}_j, \bar{y}_h \rangle + d \left( \bar{y}_h, \bar{y}_h, \bar{w}_h \right)
\]
hold for all \([ \bar{w}_h, q_h ] \in Y_h \times M_h \). Applying the results (4.10) of Lemma 4.2, we can derive that
\[
\| \bar{y}_0 - \bar{y}_{h0} \|_{1,2,\Omega} + \| \bar{p}_0 - \bar{p}_{h0} \|_{0,2,\Omega} \leq C \left( h^m + h^{m+2}_U \right),
\]
\[
\| \bar{y}_0 - \bar{y}_{h0} \|_{0,2,\Omega} \leq C \left( h^{m+1} + h^{m+2}_U \right),
\]
hold for \( 0 < h_4 \), so combining Lemma 4.3, we obtain the inequality (4.20) when \( j = 0 \). Similarly, by (2.3) one can get the proof of inequality (4.20) holds for \( 0 < h < h_{4,j} \) when \( 1 \leq j \leq M \). Therefore, set \( h_4 = \min \{ h_{4,j}, j = 0, \cdots, M \} \) and the proof is then finished. \( \square \)
Lemma 4.5. Let $\tilde{\lambda}(or \tilde{\lambda}_h)$ be the solution of Eqs. (3.4) (or Eqs. (3.15)), suppose that $\tilde{u}$ is regular, then there holds

$$|\tilde{\lambda} - \tilde{\lambda}_h| \leq C(h^{m+1} + h_{L}^{m+2} + \|\mathcal{P}_h \tilde{u} - \tilde{u}_h\|_{0,2,\Omega}).$$

(4.21)

Proof. Since $\tilde{u}$ is a regular optimal control of problem (2.1), using (2.3), (3.7) and optimality conditions (3.3), (3.14) we get that

$$\lambda_i - \lambda_{hi} = (\lambda_i - \lambda_{h_0})F_i'(\tilde{u})v_i + \lambda_{hi}F_i'(\tilde{u}_h)v_i + \lambda_{hi}(-F_i'(\tilde{u})v_i + F_i'(\tilde{u}_h)v_i)
= (\tilde{\psi}_0 - \tilde{\psi}_h) + \lambda_{hi} - \lambda_{h_0}v_i
\leq C\|v_i\|_{0,2,\Omega}(\|\tilde{\psi}_0 - \tilde{\psi}_h\|_{0,2,\Omega} + \|\tilde{\psi}_i - \tilde{\psi}_h\|_{0,2,\Omega})$$
holds for all $1 \leq i \leq M_0$. Then recall the results of Lemma 4.4, we obtain (4.21). $\square$

The following lemma follows from (3.5), (3.16), the results of Lemmas 4.4 and 4.5, whose proof only need to repeat the steps of the proof in Lemma 4.4.

Lemma 4.6. Let $[\tilde{\psi}_h, \tau_h]$ (or $[\tilde{\psi}, \tau]$) be the solution of Eq. (3.15) (or Eq. (3.4)), then

$$\|\tilde{\psi} - \tilde{\psi}_h\|_{1,2,\Omega} + \|\tau - \tau_h\|_{0,2,\Omega} \leq C(h^{m+1} + h_{L}^{m+2} + \|\mathcal{P}_h \tilde{u} - \tilde{u}_h\|_{0,2,\Omega}),$$

$$\|\tilde{\psi} - \tilde{\psi}_h\|_{0,2,\Omega} \leq C(h^{m+1} + h_{L}^{m+2} + \|\mathcal{P}_h \tilde{u} - \tilde{u}_h\|_{0,2,\Omega})$$
hold true.

Then, let us recall discrete second optimality conditions of Lemma 4.1 and the above lemmas to get the super-convergence of the control.

Lemma 4.7. Let us assume $\tilde{u}(or \tilde{u}_h)$ is the solution of Eqs. (3.4) (or Eqs. (3.15)) and conditions in Lemma 3.3 holds, then there exists $h_5$ such that

$$\|\mathcal{P}_h \tilde{u} - \tilde{u}_h\|_{0,2,\Omega} \leq C(h^{m+1} + h_{L}^{m+2})$$

(4.24)
holds for $h < h_5$.

Proof. By Lemma 4.1 and Theorem 4.1, we can choose $h_0' < h_0$ know that for $h, h_{L} < h_0'$ there hold

$$\frac{c_0}{4} \|\mathcal{P}_h \tilde{u} - \tilde{u}_h\|_{0,2,\Omega}^2
\leq \left(\frac{\partial L_h}{\partial \tilde{u}_h} (\mathcal{P}_h \tilde{u}, \tilde{\lambda}_h) - \frac{\partial L_h}{\partial \tilde{u}_h} (\tilde{u}_h, \tilde{\lambda}_h), \mathcal{P}_h \tilde{u} - \tilde{u}_h\right)
= \left(G_h(\mathcal{P}_h \tilde{u}) - \mathcal{y}_d + \sum_{i=1}^M \tilde{\lambda}_h F_i'(\mathcal{G}_h(\mathcal{P}_h \tilde{u})), G_h'(\mathcal{P}_h \tilde{u})(\mathcal{P}_h \tilde{u} - \tilde{u}_h)\right) + \alpha \left(\mathcal{P}_h \tilde{u}, \mathcal{P}_h \tilde{u} - \tilde{u}_h\right).$$
Noticing that
\[
0 = \left( \frac{\partial L}{\partial \bar{u}}(\bar{u}, \bar{\lambda}), P_h \bar{u} - \bar{u}_h \right) + \left( \bar{G}(\bar{u}) - y_d + \sum_{i=1}^{M} \bar{\lambda}_i F_i'(\bar{G}(\bar{u})), G'(\bar{u})(P_h \bar{u} - \bar{u}_h) \right) + \alpha(\bar{u}, P_h \bar{u} - \bar{u}_h)
\]
and the assumption (2.3), we know that the above inequality is equivalent to
\[
\frac{c_0}{4} \| P_h \bar{u} - \bar{u}_h \|_{0,2,\Omega}^2 \\
\leq \left( \bar{G}_h(P_h \bar{u}) - \bar{G}(\bar{u}) + \sum_{i=1}^{M} (\bar{\lambda}_i F_i'(P_h \bar{u})) - \bar{\lambda}_i F_i'(\bar{G}(\bar{u}))), \bar{G}_h'(P_h \bar{u})(P_h \bar{u} - \bar{u}_h) \right) \\
+ \left( \bar{G}(\bar{u}) - y_d + \sum_{i=1}^{M} \bar{\lambda}_i F_i'(\bar{G}(\bar{u})), \bar{G}_h'(P_h \bar{u})(P_h \bar{u} - \bar{u}_h) - G'(\bar{u})(P_h \bar{u} - \bar{u}_h) \right) \\
\leq C \left( \| \bar{G}_h(P_h \bar{u}) - \bar{G}(\bar{u}) \|_{0,2,\Omega} \| \bar{G}_h'(P_h \bar{u})(P_h \bar{u} - \bar{u}_h) \|_{0,2,\Omega} \\
+ \| \bar{G}_h'(P_h \bar{u})(P_h \bar{u} - \bar{u}_h) - G'(\bar{u})(P_h \bar{u} - \bar{u}_h) \|_{0,2,\Omega} \right) \\
+ \sum_{i=1}^{M} (\bar{\lambda}_i - \bar{\lambda}_i) \left( F_i'(\bar{G}_h(P_h \bar{u})) \bar{G}_h'(P_h \bar{u})(P_h \bar{u} - \bar{u}_h) \right).
\]
(4.25)

It is clear that the estimate
\[
\| \bar{G}_h(P_h \bar{u}) - \bar{G}(\bar{u}) \|_{0,2,\Omega} \leq C (h^{m+1} + h_U^{m_U+2})
\]
(4.26)
can be derived analogously as in Lemma 4.3 by using Lemma 4.2. Furthermore, observing the proof of Lemmas 4.4 and 4.5 and using (4.26), we obtain
\[
\| \bar{G}_h'(P_h \bar{u}) (P_h \bar{u} - \bar{u}_h) - G'(\bar{u})(P_h \bar{u} - \bar{u}_h) \|_{0,2,\Omega} \leq C (h^{m+1} + h_U^{m_U+2}) \| P_h \bar{u} - \bar{u}_h \|_{0,2,\Omega}.
\]
Substituting it and (4.26) into (4.25), we have
\[
\frac{c_0}{4} \| P_h \bar{u} - \bar{u}_h \|_{0,2,\Omega}^2 \leq C (h^{m+1} + h_U^{m_U+2}) \| P_h \bar{u} - \bar{u}_h \|_{0,2,\Omega} \\
+ \sum_{i=1}^{M} (\bar{\lambda}_i - \bar{\lambda}_i) \left( F_i'(\bar{G}_h(P_h \bar{u})) \bar{G}_h'(P_h \bar{u})(P_h \bar{u} - \bar{u}_h) \right).
\]
(4.27)

In the meantime, for all \(1 \leq i \leq M\), from the Taylor expansion of functional \(F_{hi}\) at \(P_h \bar{u}\) that there exists a real number \(s(0 < s < 1)\) and
\[
F_{hi}(\bar{u}_h) = F_{hi}(P_h \bar{u}_h) + (F_{hi}'(P_h \bar{u}_h), \bar{u}_h - P_h \bar{u}_h) + \left( \frac{1}{2} F_{hi}''(s \bar{u}_h + (1-s)P_h \bar{u}_h), (\bar{u}_h - P_h \bar{u}_h)^2 \right).
\]
we have
\[
(F_{hi}'(P_h \bar{u}), P_h \bar{u} - \bar{u}_h) = F_{hi}(P_h \bar{u}_h) - F_{hi}(\bar{u}_h) + \left( \frac{1}{2} F_{hi}''(s \bar{u}_h + (1-s)P_h \bar{u}_h), (\bar{u}_h - P_h \bar{u}_h)^2 \right).
\]
(4.28)
Then using Lemma 4.5 and the estimate \( \| \mathcal{G}_h(\mathcal{P}_h \bar{u}) - \mathcal{G} (\bar{u}) \|_{0,2 \Omega} \leq C (h^{m+1} + h_{U}^{mu+2}) \) which can be proved analogously as in Lemma 4.3, we can derive

\[
(\bar{\lambda}_{hi} - \bar{\lambda}_i)(F'_{hi}(\mathcal{P}_h \bar{u}), \mathcal{P}_h \bar{u} - \bar{u}_h) \\
= (\bar{\lambda}_{hi} - \bar{\lambda}_i)(F'_{hi}(\bar{u}) - F'_{hi}(\bar{u}_h)) + (\bar{\lambda}_{hi} - \bar{\lambda}_i)(F'_{hi}(\mathcal{P}_h \bar{u}) - F'_{hi}(\bar{u})) \\
+ (\bar{\lambda}_{hi} - \bar{\lambda}_i)(\frac{1}{2} F''_{hi}(s \bar{u}_h + (1-s) \mathcal{P}_h \bar{u}), (\bar{u}_h - \mathcal{P}_h \bar{u})^2) \\
\leq (\bar{\lambda}_{hi} - \bar{\lambda}_i)(F'_{hi}(\mathcal{P}_h \bar{u}) - F'_{hi}(\bar{u}_h)) + (\bar{\lambda}_{hi} - \bar{\lambda}_i)(\frac{1}{2} F''_{hi}(s \bar{u}_h + (1-s) \mathcal{P}_h \bar{u}), (\bar{u}_h - \mathcal{P}_h \bar{u})^2) \\
= (\bar{\lambda}_{hi} - \bar{\lambda}_i)(F(\bar{y}_h(\mathcal{P}_h \bar{u}) - F(\bar{y}_h)) + (\bar{\lambda}_{hi} - \bar{\lambda}_i)(\frac{1}{2} F''_{hi}(s \bar{u}_h + (1-s) \mathcal{P}_h \bar{u}), (\bar{u}_h - \mathcal{P}_h \bar{u})^2) \\
\leq C(\bar{\lambda}_{hi} - \bar{\lambda}_i)\| \mathcal{G}_h(\mathcal{P}_h \bar{u}) - \mathcal{G} (\bar{u}) \|_{0,2 \Omega} + (\bar{\lambda}_{hi} - \bar{\lambda}_i)(\frac{1}{2} F''_{hi}(s \bar{u}_h + (1-s) \mathcal{P}_h \bar{u}), (\bar{u}_h - \mathcal{P}_h \bar{u})^2) \\
\leq \epsilon\| \mathcal{P}_h \bar{u} - \bar{u}_h \|_{0,2 \Omega}^2 + C(\epsilon) \left\{ h^{2(m+1)} + h_{U}^{2(mu+2)} + \| \mathcal{P}_h \bar{u} - \bar{u}_h \|_{0,2 \Omega}^3 \right\}, \tag{4.29}
\]

where we also used the fact that

\[
(\bar{\lambda}_{hi} - \bar{\lambda}_i)(F'_{hi}(\bar{u}) - F'_{hi}(\bar{u}_h)) = (\bar{\lambda}_{hi} - \bar{\lambda}_i)(F(\bar{y}_h) - F(\bar{y}_h)) \leq 0.
\]

Therefore, substituting (4.29) into inequality (4.27) and using Theorem 4.1, we can select a \( h_5 < h_0 \),

\[
\| \mathcal{P}_h \bar{u} - \bar{u}_h \|_{0,2 \Omega} \leq C \left( h^{m+1} + h_{U}^{mu+2} \right)
\]

holds true for \( h, h_{U} < h_5 \).

Finally, applying Lemma 4.7 in Lemmas 4.3 - 4.6, there is

\[
| \bar{\lambda} - \bar{\lambda}_h | \leq C \left( h^{m+1} + h_{U}^{mu+2} \right), \\
\| \bar{y} - \bar{y}_h \|_{1,2 \Omega} + \| \bar{p} - \bar{p}_h \|_{0,2 \Omega} + \| \tilde{\psi} - \tilde{\psi}_h \|_{1,2 \Omega} + \| \bar{\pi} - \bar{\pi}_h \|_{0,2 \Omega} \leq C \left( h^{m+1} + h_{U}^{mu+2} \right), \\
\| \bar{y} - \bar{y}_h \|_{0,2 \Omega} + \| \tilde{\psi} - \tilde{\psi}_h \|_{0,2 \Omega} \leq C \left( h^{m+1} + h_{U}^{mu+2} \right),
\]

and

\[
\| \bar{u} - \bar{u}_h \| \leq \| \bar{u} - \bar{p} \bar{u} \|_{0,2 \Omega} + \| \bar{p} \bar{u} - \bar{u}_h \|_{0,2 \Omega} \leq C \left( h^{m+1} + h_{U}^{mu+1} \right),
\]

when \( h, h_{U} < \bar{h} := \min \{ h_i, i = 1, \ldots, 5 \} \). Therefore, combining the above inequalities, one can proof Theorem 4.2.

5 Numerical experiments

In this section, we give some numerical experiments to confirm our theoretical results derived in Section 4. In these numerical experiments, we use the software package: AFEpack, see [23] for details. We solve all the N-S equations using \( P_2-P_1 \) Hood-Taylor
elements, see [28], for instance, and approximate the optimal control and the corresponding Lagrange multiplier by the semi-smooth Newton iteration [21]. The iteration scheme is stated as follows, and the readers may find more details and the convergence proof of this type Newton-like method in [12, 30] and the references cited therein.

**Semi-smooth Newton algorithm**

**Step 1.** Set \( k = 0 \) and fix some step lengths \( \rho_i > 0 \) for \( i = 1, \ldots, M \). Select initial approximations \( \lambda^0_h, u^0_h \) and \( y^0_h, p^0_h, \psi^0_h, \tau^0_h \).

**Step 2.** Solve equations for all \( w_h \in H^0, q_h \in Q^h \):

\[
\begin{align*}
\begin{cases}
     a(y^{k+1}_h, w_h) + d(y^{k+1}_h, y^k_h, w_h) + d(y^k_h, y^{k+1}_h, w_h) + b(w_h, p^{k+1}_h) \\
     = (f + u^*_h, w_h) + d(y^k_h, y^k_h, w_h), \\
     b(y^{k+1}_h, q_h) = 0, \\
     a(\psi^{k+1}_h, w_h) + d(y^{k+1}_h, w_h, \psi^k_h) + d(y^k_h, w_h, \psi^{k+1}_h) + d(w_h, y^{k+1}_h, \psi^k_h) \\
     + d(w_h, y^k_h, \psi^{k+1}_h) - \left( (1 + \sum_{i=0}^{M} \lambda^i_h F''_i(y^k_h)) y^{k+1}_h, w_h \right) + b(w_h, \tau^{k+1}_h) \\
     = \left( \sum_{i=0}^{M} \lambda^i_h (F'_i(y^k_h) - F''_i(y^k_h) y^k_h) - y_d, w_h \right) + d(y^k_h, w_h, \psi^k_h) + d(w_h, y^k_h, \psi^k_h), \\
     b(\psi^{k+1}_h, q_h) = 0.
\end{cases}
\end{align*}
\]

**Step 3.** Let

\[
\begin{align*}
    u^{k+1}_h &= -P^h_l \psi^{k+1}_h, \\
    \lambda^{k+1}_i &= \max \left\{ 0, \lambda^k_i + \rho_i F_i(y^k_h) \right\} \quad \text{for } i = 1, \ldots, M.
\end{align*}
\]

If \( \|u^{k+1}_h - u^*_h\|_{0, \Omega} + |\lambda^{k+1}_i - \lambda^k_i| > \text{Tol} \), set \( k = k + 1 \) and then go to **Step 2**.

**Step 4.** Output \( u^{k+1}_h, \lambda^{k+1}_i, y^{k+1}_h, p^{k+1}_h, \psi^{k+1}_h, \tau^{k+1}_h \).

We investigate the problem as follows:

\[
\begin{align*}
\min_u \left\{ \frac{1}{2} \int_{\Omega} |y - y_d|^2 + \frac{1}{2} \int_{\Omega} |u|^2 \right\}, \\
\text{subject to} \\
\begin{cases}
    -\frac{1}{20} \Delta y + (y \cdot \nabla)y + \nabla p = f + u & \text{in } \Omega, \\
    \nabla \cdot y = 0 & \text{in } \Omega, \\
    y = 0 & \text{on } \partial \Omega, \\
    \|y\|_{0, 2, \Omega} \leq 1.
\end{cases}
\end{align*}
\]

(5.1)
where $\Omega = (0,1) \times (0,1)$. Those imply that $d = 2$, $\alpha = 1$, $\nu = 0.05$ and $\gamma = 1$.

In order to verify the \textit{a priori} error estimates, more precisely speaking to confirm Theorem 4.2, we perform the numerical experiments composed of three examples, in which we use different elements approximating the control in different examples on the same mesh for all the variables, respectively.

The exact solutions are set as:

$$
\begin{align*}
    y_1 &= \pi^2 \sin^2 \pi x \sin 2\pi y / C_0, \\
    y_2 &= -\pi^2 \sin^2 \pi x \sin 2\pi y / C_0, \\
    p &= 1000(\pi xy - 0.25), \\
    u_0 &= -0.25 \sin^2 \pi x \sin 2\pi y, \\
    u_1 &= 0.25 \sin 2\pi x \sin 2\pi y, \\
    \psi_0 &= 0.25 \sin^2 \pi x \sin 2\pi y, \\
    \psi_1 &= -0.25 \sin 2\pi x \sin 2\pi y, \\
    \pi &= 1000(\pi xy - 0.25), \\
    \lambda &= C_0 - 1,
\end{align*}
$$

(5.3)

where $C_0 \approx 3.8880$ such that $\|y\|_0 = 1$. For abbreviation, we denote the $L^2$-norm, $H^1$-norm or the $L^2$-norm, $H^1$-norm defined in domain $\Omega$ by $\|\cdot\|_0$, $\|\cdot\|_1$ respectively in what follows.

**Example 5.1.** Firstly, we use the quadratic elements approximating the control, namely $m = 2$, $m_U = 2$. The numerical results are listed in Table 1. The convergence rates obtained from the above results are listed in Table 2.

**Table 1:** Numerical results of Example 5.1 ($m = m_U = 2$).

<table>
<thead>
<tr>
<th>Elements</th>
<th>$[y_h, \psi_h, u_h]$ piecewise quadratic, $[p_h, \pi_h]$ linear</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mesh</td>
<td>$h = 0.2$</td>
</tr>
<tr>
<td>$|P_h u - u_h|_0$</td>
<td>2.26234195e-02</td>
</tr>
<tr>
<td>$|\psi - \psi_h|_0$</td>
<td>1.06811641e-02</td>
</tr>
<tr>
<td>$</td>
<td>\lambda - \lambda_h</td>
</tr>
<tr>
<td>$|y - y_h|_0$</td>
<td>1.19272550e-01</td>
</tr>
<tr>
<td>$|u - u_h|_0$</td>
<td>2.28117328e-02</td>
</tr>
<tr>
<td>$|\psi - \psi_h|_0$</td>
<td>5.03823596e-01</td>
</tr>
<tr>
<td>$|\pi - \pi_h|_0$</td>
<td>1.80641166e-01</td>
</tr>
</tbody>
</table>

**Table 2:** Convergent rates of Example 5.1 ($m = m_U = 2$).

| Mesh | $\|P_h u - u_h\|_0$ | $\|y - y_h\|_0$ | $\|\psi - \psi_h\|_0$ | $|\lambda - \lambda_h|$ | $\|u - u_h\|_0$ | $\|y - y_h\|_1$ |
|------|------------------|----------------|----------------|----------------|----------------|----------------|
| 1→2  | 3.9241           | 2.8533         | 3.9062         | 3.9577         | 3.9062         | 1.9835         |
| 2→3  | 4.0270           | 3.0762         | 3.9864         | 4.0294         | 3.9864         | 2.0457         |
| 3→4  | 4.0093           | 2.9968         | 3.8552         | 4.0187         | 3.8552         | 1.9970         |
Example 5.2. Secondly, we use the linear elements approximating the control, namely \( m = 2, m_U = 1 \). The numerical results are listed in Table 3. The convergence rates obtained from the above results are listed in Table 4.

<table>
<thead>
<tr>
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</tr>
</thead>
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<td>P_h u - u_h</td>
</tr>
<tr>
<td>(</td>
<td>\psi - \psi_h</td>
</tr>
<tr>
<td>(</td>
<td>\lambda - \lambda_h</td>
</tr>
<tr>
<td>(</td>
<td>y - y_h</td>
</tr>
<tr>
<td>(</td>
<td>u - u_h</td>
</tr>
<tr>
<td>(</td>
<td>y - y_h</td>
</tr>
<tr>
<td>(</td>
<td>\psi - \psi_h</td>
</tr>
<tr>
<td>(</td>
<td>p - p_h</td>
</tr>
<tr>
<td>(</td>
<td>\pi - \pi_h</td>
</tr>
</tbody>
</table>

Table 4: Convergent rates of Example 5.2.

| Mesh | \(| P_h u - u_h |_0 \) | \(| y - y_h |_0 \) | \(| \psi - \psi_h |_0 \) | \(| \lambda - \lambda_h |_0 \) | \(| u - u_h |_0 \) | \(| y - y_h |_1 \) |
|------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1→2 | 3.9744 | 2.8572 | 4.0118 | 4.0796 | 3.0451 | 1.9835 |
| 2→3 | 4.0695 | 3.0769 | 4.0088 | 4.0930 | 2.2911 | 2.0457 |
| 3→4 | 4.0108 | 2.9969 | 3.7583 | 4.0389 | 2.0424 | 1.9970 |

Example 5.3. Finally, we use the constant elements approximating the control, namely \( m = 2, m_U = 0 \). The numerical results are listed in Table 5. The convergence rates obtained from the above results are listed in Table 6.

<table>
<thead>
<tr>
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<td>P_h u - u_h</td>
</tr>
<tr>
<td>(</td>
<td>\psi - \psi_h</td>
</tr>
<tr>
<td>(</td>
<td>\lambda - \lambda_h</td>
</tr>
<tr>
<td>(</td>
<td>y - y_h</td>
</tr>
<tr>
<td>(</td>
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From the above tests in Examples 5.1-5.3, we can see the convergence rate of the control is of order 3,2,1 with respect to $m_U=2,1,0$, which coincides with (4.5) (here, $m \geq m_U$) in Theorem 4.2. The convergence rates of the state $[y_h,p_h]$ and the co-state $[\psi,\pi]$ in $H^1(\Omega) \times L^2(\Omega)$ are always of order 2, whatever we use the piecewise quadratic, the linear or the constant elements approximating the control, which are consistent with (4.4) in Theorem 4.2.

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### References


