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A Family of Characteristic Discontinuous Galerkin Methods for Transient Advection-Diffusion Equations and Their Optimal-Order L^2 Error Estimates

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Abstract. We develop a family of characteristic discontinuous Galerkin methods for transient advection-diffusion equations, including the characteristic NIPG, OBB, IIPG, and SIPG schemes. The derived schemes possess combined advantages of Eulerian-Lagrangian methods and discontinuous Galerkin methods. An optimal-order error estimate in the L^2 norm and a superconvergence estimate in a weighted energy norm are proved for the characteristic NIPG, IIPG, and SIPG scheme. Numerical experiments are presented to confirm the optimal-order spatial and temporal convergence rates of these schemes as proved in the theorems and to show that these schemes compare favorably to the standard NIPG, OBB, IIPG, and SIPG schemes in the context of advection-diffusion equations.

AMS subject classifications: 35R32, 37N30, 65M12, 76N15

Key words: Advection-diffusion equation, characteristic method, discontinuous Galerkin method, numerical analysis, optimal-order L^2 error estimate, superconvergence estimate.

1 Introduction

Transient advection-diffusion equations admit solutions with moving steep fronts and complicated structures. Classic numerical methods tend to generate numerical solutions

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with spurious oscillations, excessive numerical diffusion, or a combination of both. Since their introduction in 1970s [3, 18, 30], discontinuous Galerkin methods have become a topic of extensive research on numerically solving differential equations. These methods use piecewise polynomial trial and test functions which may be discontinuous across cell boundaries in their weak formulations and are locally mass conservative. They are inherently adaptive to local high order approximations and can capture moving steep fronts and shock discontinuities in the solution to advection-diffusion equations via the use of discontinuous approximating spaces. They are well suited for handling unstructured meshes and nonmatching grids for multidimensional problems.

Optimal-order convergence in the energy norm has been proved for a variety of primal discontinuous Galerkin methods for elliptic and parabolic differential equations [7, 17,20,22,23]. Optimal L^2 convergence has been established [22,23] for the symmetric interior penalty Galerkin (SIPG) method [30]. However, numerical experiments [7,23] reveal that the nonsymmetric discontinuous Galerkin methods, including the Oden-Babuška-Baumann (OBB) formulation [7, 17], the nonsymmetric interior penalty Galerkin (NIPG) method [19], and the incomplete interior penalty Galerkin (IIPG) method [12, 22, 23], are generally not optimal in the L^2 norm. An optimal-order error estimate in the L^2 norm has been proved for the NIPG and IIPG methods with odd degree polynomials for one-dimensional elliptic problems with a uniform partition [16].

In this paper we develop a family of characteristic discontinuous Galerkin methods for one-dimensional transient advection-diffusion equations by using an Eulerian-Lagrangian approach within a primal discontinuous Galerkin framework [4,9] (instead of the dual formulation [10, 11]). These include the characteristic SIPG method, the characteristic NIPG method, the characteristic IIPG method, and the characteristic OBB method. The developed methods retain the numerical advantages of the discontinuous Galerkin methods. Further, they stabilize the numerical approximations and generate accurate numerical solutions, even if large time steps and coarse spatial grids are used. In this paper we prove an optimal-order L^2 error estimate and a superconvergence estimate for the characteristic NIPG, SIPG, and IIPG methods. Numerical results are presented to show the convergence behavior of the proposed schemes.

The rest of the paper is organized as follows: In Section 2 we derive a reference equation satisfied by the true solution to problem (2.1). In Section 3 we develop a family of characteristic discontinuous Galerkin methods. In Section 4 we present the preliminaries used in the error analysis. Section 5 contains the main error estimate of this article. In Section 6 we present preliminary example runs to show the performance of the scheme. Section 7 contains summary and conclusions. Finally, we present the proof of auxiliary lemmas in the appendix.

2 A cell-based characteristic reference equation

We consider the initial-boundary value problem