Commun. Comput. Phys.
doi: 10.4208/cicp.2009.08.135

Vol. 7, No. 1, pp. 103-137
January 2010

Stable and Accurate Second-Order Formulation of the Shifted Wave Equation

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Received 20 August 2008; Accepted (in revised version) 4 May 2009

Communicated by Chi-Wang Shu

Available online 16 July 2009

Abstract. High order finite difference approximations are derived for a one-dimensional model of the shifted wave equation written in second-order form. The domain is discretized using fully compatible summation by parts operators and the boundary conditions are imposed using a penalty method, leading to fully explicit time integration. This discretization yields a strictly stable and efficient scheme. The analysis is verified by numerical simulations in one-dimension. The present study is the first step towards a strictly stable simulation of the second-order formulation of Einstein's equations in three spatial dimensions.

AMS subject classifications: 35L05, 35L20, 65N06, 65N12, 83C05

Key words: High-order finite difference methods, wave equation, numerical stability, second derivatives, Einstein's equations.

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1 Introduction

The present study is focused towards the numerical solution of Einstein's equations, which describe processes such as binary black holes and neutron star collisions. The outcome of this kind of simulations is considered to be crucial for the successful detection and interpretation of gravitational waves, expected to be measured by laser interferometers such as LIGO, GEO600, LISA and others. In turn, measurement of gravitational waves will constitute a strong, direct verification of Einstein's theory, and open a new window to the universe.

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In the harmonic description of general relativity, the principal part of Einstein's equations reduces to 10 curved space wave equations for the components of the space-time metric. Although these equations can be reduced to first-order symmetric hyperbolic form [7], this has the disadvantage of introducing auxiliary variables with their constraints and boundary conditions. The reduction to first-order form is also less attractive from a computational point of view considering the efficiency and accuracy [12,19]. The reasons for solving the equations on first-order form are most likely related to the maturity of CFD, which has evolved during the last 40 years. I.e., many of the stability issues for first-order hyperbolic problems have already been addressed.

Wave-propagation problems frequently require farfield boundaries to be positioned many wavelengths away from the disturbance source (for example binary black holes). To efficiently simulate these problems requires numerical techniques capable of accurately propagating disturbances (such as a gravitational wave) over long distances. It is well know that high-order finite difference methods (HOFDM) are ideally suited for problems of this type. (See the pioneering paper by Kreiss and Oliger [14]). Not all high-order spatial operators are applicable, however. For example, schemes that are G-K-S stable [9], while being convergent to the true solution as $\Delta x \rightarrow 0$, may experience non-physical solution growth in time [5], thereby limiting their efficiency for long-time simulations. Thus, it is imperative to use HOFDMs that do not allow growth in time; a property termed "strict stability" [8]. Deriving strictly stable, accurate and conservative HOFDM is a significant challenge that has received considerable past attention. (For example, see references [1,3,10,11,16,31–33,38]).

The energy method (see for example [8]) is a common technique to derive well-posedness for initial-boundary value problems. A very powerful way of obtaining provable strictly stable numerical approximations is to mimic the underlying continuous energy estimate. A well-proven HOFD methodology that ensures this is the summation-by-parts simultaneous approximation term (SBP-SAT) method. The SBP-SAT method simply combines finite difference operators that satisfy a summation-by-parts (SBP) formula [13], with physical boundary conditions implemented using either the Simultaneous Approximation Term (SAT) method [5], or the projection method [19,28,29]. Examples of the SBP-SAT approach can be found in references [6,15,17,18,20,22,24–27,34,35].

Deriving strictly stable numerical simulations of Einstein's equations on second-order form has proven to be a very difficult task [2,4,23,37], especially for HOFDMs. In the present study this situation is illustrated by the shifted wave equation in 1-D that captures most of the stability issues without introducing unnecessary complications. This 1-D problem was analyzed in [37] for a second-order accurate approximation.

For the Einstein's equations (and the shifted wave equation) written on second-order form, the regular energy estimate fails in the most interesting applications, which required the introduction of a modified energy estimate. The existing SBP-SAT method (here referred to as the *standard* SBP-SAT method) is based on the regular energy estimates, which means that the *standard* SBP-SAT method has to be modified in order to fit the mentioned modified energy estimate.