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A Simple, Fast and Stabilized Flowing Finite Volume Method for Solving General Curve Evolution Equations

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Abstract. A new simple Lagrangian method with favorable stability and efficiency properties for computing general plane curve evolutions is presented. The method is based on the flowing finite volume discretization of the intrinsic partial differential equation for updating the position vector of evolving family of plane curves. A curve can be evolved in the normal direction by a combination of fourth order terms related to the intrinsic Laplacian of the curvature, second order terms related to the curvature, first order terms related to anisotropy and by a given external velocity field. The evolution is numerically stabilized by an asymptotically uniform tangential redistribution of grid points yielding the first order intrinsic advective terms in the governing system of equations. By using a semi-implicit in time discretization it can be numerically approximated by a solution to linear penta-diagonal systems of equations (in presence of the fourth order terms) or tri-diagonal systems (in the case of the second order terms). Various numerical experiments of plane curve evolutions, including, in particular, nonlinear, anisotropic and regularized backward curvature flows, surface diffusion and Willmore flows, are presented and discussed.

AMS subject classifications: 35K65, 65N40, 53C80, 35K55, 53C44, 65M60

Key words: Geometric partial differential equations, evolving plane curves, mean curvature flow, anisotropy, Willmore flow, surface diffusion, finite volume method, semi-implicit scheme, tangential redistribution.

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1 Introduction

The main purpose of this paper is to propose a simple, fast and stable Lagrangian method for computing evolution of closed smooth embedded plane curves driven by a normal velocity of the form $\beta(\partial_s^2 k, k, \nu, x)$ which may depend on the intrinsic Laplacian $\partial_s^2 k$ of the curvature k, on the curvature k itself, on the tangential angle ν and the curve position vector x. We shall restrict our attention to the following form of the normal velocity:

$$\beta = -\delta\, \partial_s^2 k + b(k,\nu) + F(x). \tag{1.1}$$

Here $\delta \geq 0$ is a constant and $b = b(k,\nu)$ is a smooth function satisfying $b(0,\cdot) = 0$. If $\delta > 0$ then there is no constraint on the monotonicity of b with respect to k. On the other hand, if $\delta = 0$, then we shall assume the function b is strictly increasing with respect to the curvature.

There are many interesting evolutionary models in various applied fields of science, technology and engineering that can be described by geometric Eq. (1.1). For example, putting $\delta = 0$ we obtain the normal velocity $\beta = b(k,\nu) + F(x)$ representing the well-known anisotropic mean curvature flow arising in the motion of material interfaces during solidification and in affine invariant shape analysis (see, e.g., [1,9,10,15,16,22,23,32]). The term $F(x)$ represents an external driving force, like e.g., a given velocity field projected to the normal vector of a curve, or any other constant or scalar function depending on the current curve position. It drives the curve in the inner (if $F(x) > 0$) or outer (if $F(x) < 0$) normal direction. In the case $\delta = 1$ two well-known examples arise when studying a motion of the so-called elastic curves. It is the surface diffusion, in the case $b = 0$, and the Willmore flow where $b = -\frac{1}{2}k^3$. The surface diffusion is often used in computational fluid dynamics and material sciences, where the encompassing area of interface should be preserved. The case when $b = -\frac{1}{2}k^3$ arises from the model of the Euler-Bernoulli elastic rod — an important problem in structural mechanics [6,11,12]. The evolutionary models having the normal velocity of the form (1.1) are often adopted in image segmentation where elastic and geodesic curves are used in order to find image objects in an automatic way [7,19,20,27,29]. By our method we are able to handle a regularized backward mean curvature flow in which b is a decreasing function of the curvature k like e.g. $b(k,\nu) = -k$. We regularize the backward mean curvature flow by adding a small fourth order diffusion term $0 < \delta \ll 1$. To our knowledge, first experiments of this kind are presented in this paper.

The main idea of our approach is based on accompanying the geometric equation (1.1) by a stabilizing tangential velocity and in rewriting it into a form of intrinsic an partial differential equation (PDE) for the curve position vector. The resulting PDE contains fourth, second and first order spatial differential terms that are approximated by means of the flowing finite volume method [25]. For time discretization we follow semi-implicit approach leading to a solution to a linear system of equations at each time level. That can be done efficiently, and, due to tangential stabilization, we hope that this direct Lagrangian