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Short Note

On the Use of Adjoint-Based Sensitivity Estimates to Control Local Mesh Refinement

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Abstract. The goal of efficient and robust error control, through local mesh adaptation in the computational solution of partial differential equations, is predicated on the ability to identify in an *a posteriori* way those localized regions whose refinement will lead to the most significant reductions in the error. The development of *a posteriori* error estimation schemes and of a refinement infrastructure both facilitate this goal, however they are incomplete in the sense that they do not provide an answer as to where the maximal impact of refinement may be gained or what type of refinement — elemental partitioning (h-refinement) or polynomial enrichment (p-refinement) — will best lead to that gain. In essence, one also requires knowledge of the sensitivity of the error to both the location and the type of refinement. In this communication we propose the use of adjoint-based sensitivity analysis to discriminate both where and how to refine. We present both an adjoint-based and an algebraic perspective on defining and using sensitivities, and then demonstrate through several one-dimensional model problem experiments the feasibility and benefits of our approach.

AMS subject classifications: 65M60, 65M70, 65N50

Key words: Adaptivity, error estimation, hp-finite elements, sensitivity, discrete adjoint.

1 Introduction

The use of adaptivity is widely accepted as an essential component in the efficient and reliable implementation of finite element (FE) algorithms for the solution of a wide range of partial differential equations (PDEs) [4, 12]. In particular, for problems whose solution exhibits steep fronts or sharp layers (*e.g.* boundary layers), so-called h-refinement

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is usually an appropriate strategy for producing a mesh size that is of the same order as the feature in question [3, 9, 10]. Conversely, once the mesh size is suitable, polynomial enrichment (also known as p-refinement) is generally the most accurate and cost effective form of refinement [1, 11]. In the last three decades, as practitioners have sought to develop algorithms that both incorporate and drive these adaptive procedures fully automatically, there has been an enormous research focus on the development of reliable and accurate *a posteriori* error estimates [2–5]. These estimates typically provide information on the total error of a computed solution, the distribution of this error throughout the domain and/or information on the error in some derived solution-dependent quantity. Traditionally, this information has been used both to decide when a computed solution is of sufficient accuracy for no further calculations to be required and, in the case where the accuracy is deemed to be insufficient, to decide how to adapt the FE trial space. Typically, the approach used is to refine in those regions where the estimated error is the greatest: either by refining those elements whose error is within a certain percentage of the total [3], or refining the elements with the greatest error until the cumulative total is within a given percentage of the estimated error [12]. The criteria for deciding whether this refinement should be in h or p is rather more varied but is typically based upon some form of estimate as to which is likely to be the most beneficial [1, 8].

This short communication introduces a new approach for controlling local adaptivity within an hp-finite element code. The goal is to explore whether it is possible to use more information from the estimated error for deciding both where to adapt and/or how to adapt. The approach is based upon the assumption that we have a reliable *a posteriori* error estimate (E say), that is easily computable, and then to attempt to compute the sensitivity of this estimate to the addition of further p- or h- degrees of freedom. A standard adjoint approach is used to compute these sensitivities efficiently and it is demonstrated that an adaptive strategy based upon these values can have advantages over other, more traditional, approaches.

2 Notation and formulation

Consider as our model problem a linear second-order two-point boundary value problem (BVP) of the form:

$$\frac{d}{dx}\left(a(x)\frac{du}{dx}\right)+b(x)\frac{du}{dx}+c(x)=0 \tag{2.1}$$

subject to Dirichlet boundary conditions on the domain (x_0,x_N), where $a(x)>0$. Suppose the domain is divided into N intervals, $x_0<x_1<\cdots<x_{N-1}<x_N$, and let $\{\phi_0^1,\cdots,\phi_N^1\}$ be the usual basis (of local hat functions) for the space of continuous piecewise polynomials of degree one on this mesh. For simplicity assume that $u(x_0)=u(x_N)=0$, so the corresponding piecewise linear FE trial function takes the form:

$$u^1(x)=\sum_{i=1}^{N-1}u_i^1\phi_i^1(x), \tag{2.2}$$