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# Direct Vlasov Solvers with High-Order Spectral Element Method

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**Abstract.** This paper presents the development of parallel direct Vlasov solvers using the Spectral Element Method (SEM). Instead of the standard Particle-In-Cell (PIC) approach for kinetic space plasma simulation, i.e. solving the Vlasov-Maxwell equations, the direct method has been used in this paper. There are several benefits to solve the Vlasov equation directly, such as avoiding noise associated with the finite number of particles and the capability to capture the fine structure in the plasma, etc. The most challenging part of direct Vlasov solver comes from high dimension, as the computational cost increases as $N^{2d}$, where d is the dimension of the physical space. Recently due to fast development of supercomputers, the possibility of high dimensions becomes more realistic. A significant effort has been devoted to solve the Vlasov equation in low dimensions so far, now more interests focus on higher dimensions. Different numerical methods have been tried so far, such as finite difference method, Fourier spectral method, finite volume method, etc. In this paper SEM has been successfully applied to construct these solvers. SEM has shown several advantages, such as easy interpolation due to local element structure and long time integration due to its high order accuracy. Domain decomposition in high dimensions have been used for parallelization, these include scalable parallel 1D and 2D Poisson solvers. Benchmark results have been shown and simulation results have been reported for two different cases: one dimension (1P1V), and two dimensions (2P2V) in both physical and velocity spaces.

**AMS subject classifications**: 68U20, 68W10, 65C20, 65Z05

**Key words**: Vlasov equation, Spectral Element Method, Poisson's equation, interpolation, domain decomposition.

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# 1 Introduction

Plasma and charged particle simulations have great importance in science. There are three different approaches to simulate plasma: the microscopic model, the kinetic model and the fluid model. The microscopic model is governed by Liouville's equation. Each charged particle is described by 6 variables $(x,y,z,v_x,v_y,v_z)$, for a system of $N$ particles, there are 6N variables in total. This requires to solve the Liouville equation in a phase space with 6N dimensions, which obviously exceeds the capability of current supercomputers. On the other hand, the simplest model is the fluid model, which treats the plasma as a conducting fluid with electromagnetic force exerting on it. This leads to solving the Magnetohydrodynamics (MHD) equations in 3D ($x$, $y$ and $z$). MHD solves for the average quantities, such as density and charge, which makes it difficult to describe fine structures in the plasma. Due to computer speed limitations, MHD is a realistic approach in plasma simulation. Between these two models is the kinetic model, which solves the charge density function by solving Boltzmann or Vlasov equations in 6 dimensions $(x,y,z,v_x,v_y,v_z)$. The Vlasov equation describes the evolution of a system of particles under the effects of self-consistent electromagnetic fields. This paper deals with the kinetic model. There are two different ways to solve the kinetic model. The most popular one is to represent the beam bunch by macro particles and push the macro particles along the characteristics of Vlasov equation. This is the so called Particle-In-Cell (PIC) method [3], which utilizes the motion of the particles along the characteristics of the Vlasov-equation using a Lagrange-Euler approach. In principle it simplifies the Partial Differential Equation (PDE) to an Ordinary Differential Equation (ODE). The interaction between charged particles, which is called space charge effect, is accounted for by first distributing charges on a fixed grid, then solving Poisson's equation. Then the electric field from the potential solution can be computed. The PIC method has the advantages of fast speed and easy implementation, but similar to MHD, it is hard to describe fine structures in the plasma. Furthermore, there is noise associated with the finite number of particles used in the simulation, and this noise decreases very slowly as $1/\sqrt{N}$ when the number of particles N is increased. The other way to solve the kinetic model is to solve the Vlasov equation directly. This can overcome the shortcomings of the PIC method, but due to the high dimensional nature of the Vlasov equation, numerical simulations have only been conducted in low dimensions such as 1P1V or the axisymmetic case. Recently, 2P2V simulations have been reported [15]. As the speed and memory of supercomputers increases, 2P2V problems can be easily solved and higher dimensions could be realized in the near future.

Since Cheng and Knorr published their original, ground-breaking paper on a fixed mesh in the phase space [7], many different numerical methods have been adopted to solve the Vlasov equation. Generally, they can be divided into the Fourier spectral method, the finite element method, the finite difference method, the finite volume method, and the semi-Lagrangian method.

The Fourier spectral method has been used for domains with periodic boundary con-