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Convergence Analysis for Stochastic Collocation Methods to Scalar Hyperbolic Equations with a Random Wave Speed

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Abstract. For a simple model of a scalar wave equation with a random wave speed, Gottlieb and Xiu [Commun. Comput. Phys., 3 (2008), pp. 505-518] employed the generalized polynomial chaos (gPC) method and demonstrated that when uncertainty causes the change of characteristic directions, the resulting deterministic system of equations is a symmetric hyperbolic system with both positive and negative eigenvalues. Consequently, a consistent method of imposing the boundary conditions is proposed and its convergence is established under the assumption that the expansion coefficients decay fast asymptotically. In this work, we investigate stochastic collocation methods for the same type of scalar wave equation with random wave speed. It will be demonstrated that the rate of convergence depends on the regularity of the solutions; and the regularity is determined by the random wave speed and the initial and boundary data. Numerical examples are presented to support the analysis and also to show the sharpness of the assumptions on the relationship between the random wave speed and the initial and boundary data. An accuracy enhancement technique is investigated following the multi-element collocation method proposed by Foo, Wan and Karniadakis [J. Comput. Phys., 227 (2008), pp. 9572-9595].

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1 Introduction

Recently there has been a growing interest in designing efficient methods for the solution of ordinary/partial differential equations with random inputs. The methods include

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Monte Carlo and sampling based methods, perturbation methods, operator based methods and the generalized polynomial chaos (gPC) method, see, e.g., [1,6,13,14]). Among these methods, the gPC method has become one of the most widely used methods. With gPC, stochastic solutions are expressed as orthogonal polynomials of the input random parameters, and different types of orthogonal polynomials can be chosen to achieve better convergence. It is essentially a spectral representation in random space, and exhibit fast convergence when the solution depends smoothly on the random parameters.

Although the polynomial chaos methods (and gPC method) have been extensively applied to analyze PDEs that contain uncertainties, this approach is rarely applied to hyperbolic systems. Gottlieb and Xiu [7] made the first attempt by considering a simple model of a scalar wave equation with random wave speeds. It was shown that when uncertainty causes the change of characteristic directions, the resulting deterministic system of equations is a symmetric hyperbolic system with both positive and negative eigenvalues. A consistent method of imposing the boundary conditions is proposed. A numerical method based on the gPC method is introduced and its convergence theory is established.

In this work, we also consider the same model scalar wave equation with random wave speed using the stochastic collocation methods. Collocation methods have been studied and used in different disciplines for uncertainty quantification (see, e.g., Tatang [11], Xiu and Hesthaven [15], Keese and Matthies [8] Ganapathysubramanian and Zabaras [5]). In collocation methods one seeks to satisfy the governing differential equations at a discrete set of points, called "nodes", in the corresponding random space. Two of the major approaches of high-order stochastic collocation methods are the Lagrange interpolation approach, see Xiu and Hesthaven [15] and later (independently) in [1], and the pseudo-spectral gPC approach from [13].

Following the methods introduced by Tatang [11], we use the roots of the next higher order polynomial as the points at which the approximation is to be found. Let $\Theta = \{y_k\}_{k=1}^N \in \Gamma$ (the parameter space) be such a set of nodes, where N is the number of nodes. A Lagrange interpolation of the solution $w(x,y)$ can be written as

$$I^N w(x,y) = \sum_{k=1}^{N} \tilde{w}_k(x) F_k(y), \tag{1.1}$$

where

$$F_k \in \mathcal{P}_N, \quad F_i(y_k) = \delta_{ik}, \quad 1 \le i,k \le N, \tag{1.2}$$

are the Lagrange interpolation polynomials, and $\tilde{w}_k(x) := w(x,y_k)$, $1 \le k \le N$, is the value of w at the given node $y_k \in \Theta$.

In this work, we will apply the Lagrange interpolation approach to the model scalar wave equation with random wave speed (see, [7]):

$$\partial_t u(x,t;y(\omega)) = c(y(\omega)) \partial_x u(x,t;y(\omega)), \quad x \in D \equiv (-1,1), \quad t > 0, \tag{1.3}$$

$$u(x,0;y(\omega)) = u_0(x;y(\omega)), \qquad x \in D. \tag{1.4}$$