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A Technique for High-Order Treatment of Diffusion Terms in Semi-Lagrangian Schemes

Roberto Ferretti*

Dipartimento di Matematica, Università di Roma Tre, L.go S. Leonardo Murialdo, 1, 00146 Roma, Italy.

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Abstract. We consider in this paper a high-order, semi-Lagrangian technique to treat possibly degenerate advection-diffusion equations, which has been proposed in similar forms by various authors. The scheme is based on a stochastic representation formula for the solution, which allows to avoid the splitting between advective and diffusive part of the evolution operator. A general theoretical analysis is carried out in the paper, with a special emphasis on the possibility of using large Courant numbers, and numerical tests in one and two space dimensions are presented.

AMS subject classifications: 65M25, 65M12, 60H30

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1 Introduction

This paper is devoted to a semi-Lagrangian type treatment of second order terms in advection-dominated, possibly degenerate, parabolic equations. Although several extensions are possible, we will use the advection-diffusion equation,

$$\begin{cases} v_t(x,t) = \sum_{i,j=1}^{N} a_{ij} \dfrac{\partial^2}{\partial x_i \partial x_j} v(x,t) + f(x)\cdot\nabla v(x,t) + g(x), \\ v(x,0) = v_0(x), \end{cases} \tag{1.1}$$

(with $x\in\mathbb{R}^N$, $t\in[0,T]$, v_0 compactly supported in $\mathbb{R}^N$) as a model problem to describe the technique and to carry out a general convergence analysis. Here, we assume that

*Corresponding author. *Email address:* `ferretti@mat.uniroma3.it` (R. Ferretti)

$A=(a_{ij})\in\mathbb{R}^{N\times N}$ is a positive semidefinite matrix, thus including degenerate second-order operators. In order to have more explicit results, we will possibly assume in the sequel that the advection term is driven by a constant vectorfield:

$$f(x)\equiv\begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}. \tag{1.2}$$

The semi-Lagrangian (SL) schemes stem from the so-called Courant-Isaacson-Rees scheme (see [5]), and at the moment they are very popular in the Numerical Weather Prediction community. In this setting, they have been introduced by Wiin-Nielsen in [26] and brought to the present form by Robert in the 80s (see [23] and the review paper [25]). In a partly independent way, SL schemes have also been proposed in [4] and widely applied to plasma physic problems since (see, e.g., [14,24]).

The general idea of SL methods is to reconstruct the solution by integrating numerically the equation along the characteristics starting from any grid point, not over the whole time interval (as it would be the case in the particle method), but over a single time step. The scheme is constructed by coupling a numerical method for ODEs (to compute the upwind points with respect to the grid nodes) with an interpolation formula (to recover the value of the solution in such points, which are not in general grid points themselves). The comparison with more classical Eulerian difference schemes shows that in general SL schemes have a higher computational cost per time step, but that they also allow for larger time steps.

If $A=0$ (i.e. in the case of pure advection), the schemes rely on the representation formula

$$v(x,t)=\int_0^t g(y(s))ds+v_0(y(t)), \tag{1.3}$$

where $y(t)$ is the solution of

$$\begin{cases} \dot{y}(s)=f(y(s)), \\ y(0)=x. \end{cases} \tag{1.4}$$

Although using the representation formula (1.3) is very natural in the case of (linear) first order equations, the situation gets more complex when a diffusion term appears. In this situation, the usual response is to split the evolution operator and treat in semi-Lagrangian way only the first order part. This results either in very severe time-step bounds, or in the additional computational effort of solving an implicit scheme for the second order term, with the further drawback that the splitting itself introduces a limitation in the consistency rate of the scheme. A second approach is related to the so-called Lagrange-Galerkin schemes (proposed independently in [7] and [22]) which however cannot be exactly implemented in general (see [21]).

On the other hand, a natural extension of the technique used for pure advection equations can be provided by the stochastic representation (Feynman-Kac) formula for the