East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.281018.100419

Vol. **10**, No. 1, pp. 1-21
February 2020

TTSCSP-Based Iteration Methods for Complex Weakly Nonlinear Systems

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Received 28 October 2018; Accepted (in revised version) 10 April 2019.

Abstract. We present Picard-TTSCSP and nonlinear TTSCSP-like iteration methods for systems of weakly nonlinear equations. These methods require solving two subsystems with constant positive definite coefficient matrices. Such subsystems are solved by the conjugate gradient method. Local convergence of the methods is established and numerical experiments demonstrate the efficiency of the methods.

AMS subject classifications: 65F10, 65F50, 65W05

Key words: Weakly nonlinear system, TTSCSP iteration method, inner/outer iteration scheme.

1. Introduction

Let $A \in \mathbb{C}^{n\times n}$ be a large, sparse, complex symmetric matrix, $\mathbb{D} \subset \mathbb{C}^n$ and $\phi : \mathbb{D} \to \mathbb{C}^n$ a continuously differentiable function. We consider the iterative solution of the systems of weakly nonlinear equations

$$Au = \phi(u) \text{ or equivalently } F(u) = Au - \phi(u) = 0, \tag{1.1}$$

with a matrix $A = W + iT$, where $W \in \mathbb{R}^{n\times n}$ and $T \in \mathbb{R}^{n\times n}$ are symmetric positive definite and symmetric positive semidefinite matrices, respectively. The system (1.1) is called weakly nonlinear if the linear term Au is strongly dominant over the term $\phi(u)$ in a norm [14, 30]. Such systems arise in various areas of scientific computing and engineering, including the discretisation of nonlinear partial differential equations [3,4,12,13,24], collocation of nonlinear integral equations [28], saddle point problems in image processing [6,15] and other applications [20].

The Newton iteration method [2, 17] is an efficient tool for solving systems of nonlinear equations (1.1). However, at each iteration step, the method uses a Jacobian matrix, the construction of which is costly and complicated. Various approaches to improve the efficiently of the method have been proposed — cf. Refs. [2–5, 7, 11, 14, 16, 18, 29]. In

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particular, recent work of Bai *et al.* [10], where Hermitian and skew-Hermitian splitting (HSS) iteration methods [9] are used as inner solvers for the Newton's method, introduces a new class of Newton-HSS methods for large sparse systems of nonlinear equations. Bai [3] proposed sequential two-stage iteration methods for nonlinear equations (1.1) — cf. also Refs. [5, 11]. Picard-HSS and nonlinear HSS-like iteration methods are developed in [14]. Zhu *et al.* [32] considered Picard-CSCS and nonlinear CSCS-like iteration methods for weakly nonlinear systems. Using the MHSS iteration [8] as an inner solver for the Picard method, Yang *et al.* [31] developed Picard-MHSS and the nonlinear MHSS-like iteration methods. These methods can be described as follows.

Picard-MHSS iteration method. Given an initial guess $u^{(0)} \in \mathbb{D}$ and a sequence $\{l_k\}_{k=0}^{\infty}$ of positive integers, for $k = 0, 1, 2, \ldots$ compute $u^{(k+1)}$ by the iteration scheme below until $\{u^{(k)}\}$ satisfies a stopping criterion:

(a) Set $u^{(k,0)} := u^{(k)}$.

(b) To obtain $u^{(k,l+1)}$ for $l = 0, 1, 2, \ldots, l_k - 1$, solve the following linear system:

$$\begin{aligned}(\alpha I + W)u^{(k,l+1/2)} &= (\alpha I - iT)u^{(k,l)} + \phi(u^{(k)}),\\ (\alpha I + T)u^{(k,l+1)} &= (\alpha I + iW)u^{(k,l+1/2)} - i\phi(u^{(k)}),\end{aligned}$$

where α and β are given positive constants.

(c) Set $u^{(k+1)} := u^{(k,l_k)}$.

Following the works [14, 31], Li *et al.* [21] used the lopsided PMHSS (LPMHSS) iterations [22] as an inner solver for the Picard method and developed Picard-LPMHSS and nonlinear LPMHSS-like iteration methods.

Picard-LPMHSS iteration method. Given an initial guess $u^{(0)} \in \mathbb{D}$ and a sequence $\{l_k\}_{k=0}^{\infty}$ of positive integers, for $k = 0, 1, 2, \ldots$ compute $u^{(k+1)}$ by the iteration scheme below until $\{u^{(k)}\}$ satisfies a stopping criterion:

(a) Set $u^{(k,0)} := u^{(k)}$.

(b) For $l = 0, 1, 2, \ldots, l_k - 1$, solve the following linear system to obtain $u^{(k,l+1)}$:

$$\begin{aligned}Wu^{(k,l+1/2)} &= -iTu^{(k,l)} + \phi(u^{(k)}),\\ (\alpha P + T)u^{(k,l+1)} &= (\alpha P + iW)u^{(k,l+1/2)} - i\phi(u^{(k)}),\end{aligned}$$

where α is a given positive constant and P a symmetric positive definite matrix.

(c) Set $u^{(k+1)} := u^{(k,l_k)}$.

Recently, Salkuyeh and Siahkolaei [26] proposed a two-parameter two-step scale-splitting (TTSCSP) method for systems of linear equations

$$(W + iT)x = b,$$