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A Space-Time Petrov-Galerkin Spectral Method for Time Fractional Fokker-Planck Equation with Nonsmooth Solution

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Abstract. An STPG spectral method for TFFP equations with nonsmooth solutions is developed. The numerical scheme is based on generalised Jacobi functions in time and Legendre polynomials in space. The generalised Jacobi functions match the leading singularity of the corresponding problem. Therefore, the method performs better than methods with polynomial bases. The stability and convergence of the method are proved. Numerical experiments confirm the theoretical error estimates.

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Key words: Time fractional Fokker-Planck equation, nonsmooth solution, generalised Jacobi functions, space-time Petrov-Galerkin spectral method, error estimate.

1. Introduction

We consider the time fractional Fokker-Planck (TFFP) equation [22, 23]

$$\begin{aligned}&\partial_t u = {}_0^R D_t^{1-\alpha}\big\{\partial_x[p(x)u(x,t)] + K_\alpha \partial_{xx} u(x,t) + f(x,t)\big\},\\ &(x,t)\in(a,b)\times(0,T]\end{aligned} \tag{1.1}$$

with the initial and boundary conditions

$$\begin{aligned}&u(x,0)=u_0(x), \quad x\in(a,b),\\ &u(a,t)=u_a(t), \quad u(b,t)=u_b(t), \quad t\in I:=[0,T],\end{aligned}$$

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where $u(x,t)$ is a probability density function, $p(x)$ a nonpositive and monotonically decreasing function in the interval $[a,b]$, $K_\alpha > 0$ a diffusion constant and ${}_0^R D_t^{1-\alpha}$, $0 < \alpha \leq 1$ the left Riemann-Liouville fractional derivative defined by

$$ {}_0^R D_t^{1-\alpha} u(t) := \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_0^t \frac{u(s)}{(t-s)^{1-\alpha}} ds, \quad t \in I. $$

If $\alpha = 1$, the Eq. (1.1) becomes the classical Fokker-Planck (FP) equation. However, classical FP equations do not properly describe anomalous diffusion processes in highly non-homogeneous medium spaces [3, 24]. Instead, fractional Fokker-Planck (FFP) equations should be used. For example, space fractional Fokker-Planck equations are adopted to describe Lévy flights, TFFP equations to characterise the traps, and the space and time fractional Fokker-Planck (STFFP) equations handle competition between Lévy flights and traps — [1, 2, 9, 11, 15, 16, 19, 24, 25].

Lately, numerical methods for the Eq. (1.1) attracted considerable attention. Thus finite difference methods are discussed in [4, 5, 7, 13, 31, 32] and finite element methods and finite volume methods in [8,14,17,18,36]. Recently Zheng *et al.* [35] proposed a space-time spectral method based on Jacobi polynomials for temporal discretisation and on Fourier-like basis functions for spatial discretisation, whereas Yang *et al.* [33] suggested a spectral collocation method based on both temporal and spatial discretisations with a spectral expansion of Jacobi interpolation polynomials. For the STFFP equation, Zhang *et al.* [34] employed a time-space spectral method with Jacobi polynomials for temporal discretisation and Legendre polynomials for spatial discretisation. A pseudospectral method was discussed in [12, 30].

It is worth noting that polynomial approximations are not efficient in the case of Caputo or Riemann-Liouville fractional differential equations because of initial or endpoint singularities in the solutions. As is shown in [33–35], the methods using Jacobi polynomials in the discretisation of time fractional derivatives, may not be of the highest accuracy — viz. these methods fail to achieve spectral accuracy when solutions are not smooth. Nevertheless, Chen *et al.* [6] showed that the Petrov-Galerkin method using generalised Jacobi functions (GJFs) is efficient for a class of prototypical fractional initial value problems and fractional boundary value problems of general order.

The aim of this work is to study a space-time Petrov-Galerkin (STPG) spectral method for TFFP equations with nonsmooth solutions. In order to match the singularities in the corresponding solutions, time fractional derivatives can be approximated by suitable GJFs [6,28,29], while Legendre polynomials are employed for space approximations [26]. Moreover, we also analyse the errors of the method proposed.

The remainder of this paper is organised as follows. In Section 2, equivalent equations for the Eq. (1.1) are provided and some functional spaces as well as projection operators are defined. In Section 3, we consider a Petrov-Galerkin method for the TFFP equation. Section 4 deals with error estimates for the STPG spectral scheme. Numerical results, presented in Section 5, support theoretical findings and show the effectiveness of the scheme. Our conclusions are in the last section.