

A Maximum-Entropy Meshfree Method for Computation of Invariant Measures

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Abstract. Let $S : X \rightarrow X$ be a nonsingular transformation such that the corresponding Frobenius-Perron operator $P_S : L^1(X) \rightarrow L^1(X)$ has a stationary density f^* . We propose a maximum-entropy method based on a meshfree approach to the numerical recovery of f^* . Numerical experiments show that this approach is more accurate than the maximum-entropy method based on piecewise linear functions, provided that the moments involved are known. Moreover, it has a smaller computational cost than the method mentioned.

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1. Introduction

Many problems in science and engineering are related to the asymptotic properties of discrete dynamical systems. In particular, the evolution of density functions generated by the discrete dynamical systems has a special importance.

The maximum-entropy method proposed by Jaynes [10] found various applications in computing invariant measures [7, 8], calculation of Lyapunov exponents of chaotic mappings [9] and solving Fredholm integral equations [11, 13]. The traditional maximum-entropy method uses standard monomials $1, x, x^2, \dots, x^n$ as moment functions and leads to highly ill-conditioned nonlinear equations, which are difficult to solve. The number of the moments is thus restricted. Recently, Ding *et al.* [8] proposed a maximum-entropy method based on piecewise linear basis functions. The method overcomes the shortcoming of ill conditioning. Theoretical analysis and numerical results show that the maximum-entropy method based on piecewise linear basis functions has an error rate $\mathcal{O}(1/n)$ when

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the number of moments increases. This is better than the error rate $\mathcal{O}(\ln n/n)$ of the famous Ulam method [16].

Meshfree numerical methods have certain advantage over finite element methods because they only exploit nodal information. This overcomes the difficulty with mesh construction, especially in high dimensional problems. In addition, such methods have a high accuracy.

In this work, we approximate the invariant measure of a chaotic mapping by a new maximum-entropy method, which is based on the meshfree approach and uses basis functions from the local maximum-entropy approximation scheme in [1]. The local maximum-entropy approximation scheme is a seamless bridge between finite elements and meshfree methods. The basis functions have some similarities with the ones in the moving least squares method [2, 3, 15]. At the same time, they demonstrate important advantages. The basis functions satisfy partition of unity and approximate local support properties. Therefore, the corresponding Jacobian matrix of the nonlinear equations obtained by the maximum-entropy method is banded and positive definite. This guaranties that the nonlinear equations have a unique solution and can be solved efficiently.

The outline of this paper is as follows. Preliminaries are provided in Section 2. The moment functions of the meshfree method are given in Section 3. In Section 4 we introduce a maximum-entropy meshfree method and study its convergence using the results in [4–6]. Section 5 illustrates the algorithm by several examples and our conclusions are in Section 6.

2. Frobenius-Perron Operators and Maximum Entropy

The dynamics of various processes often exhibits a complicated behavior. A single solution of a dynamical system over a long period of time obtained by the asymptotic behavior of the trajectory of the system, is generally less useful. Such long-time behavior can be better described by estimating the probability that a domain is hit. As time approaches to infinity, the probability approaches to a limit, which is the distribution of the trajectory. Various mapping properties can be studied by the distribution or density. The density can be considered as an ensemble of initial points in the phase space. In this case, instead of considering the asymptotic behavior of individual points in the phase space, we use the Frobenius-Perron operator to study the density evolution.

Definition 2.1. Let (X, Σ, μ) be a σ -finite measure space. A transformation $S : X \rightarrow X$ is called nonsingular if the condition $\mu(B) = 0$ yields $\mu(S^{-1}(B)) = 0$ and measure preserving if $\mu(S^{-1}(B)) = \mu(B)$ for all $B \in \Sigma$. In the later case, the measure μ is said to be invariant under S .

Definition 2.2. A linear operator $P_S : L^1(X) \rightarrow L^1(X)$ such that

$$\int_B P_S f(\mathbf{x}) d\mathbf{x} = \int_{S^{-1}(B)} f(\mathbf{x}) d\mathbf{x}$$

for all $B \in \Sigma$, is called the Frobenius-Perron operator associated with S .